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On approximating the solution of the non-stationary Stokes equations using the cell discretization algorithm

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Abstract

The cell discretization algorithm, a nonconforming extension of the finite element method, is used to obtain approximations to the velocity and pressure satisfying the nonstationary Stokes equations. Error estimates show convergence of the approximations. An implementation using polynomial bases is described that permits the use of the continuous approximations of the h–p finite element method and exactly satisfies the solenoidal requirement. We express the error estimates in terms of the diameter h of a cell and the degree p of the approximation on each cell. Results of an experiment with $p \leqslant 10$ are presented that confirm the theoretical estimates. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

The nonstationary Stokes problem is to find a pair $\langle \mathbf{u}(x,t), \Pi(x,t)\rangle$ satisfying

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \operatorname{grad} \Pi = \mathbf{f}, \tag{1}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{2}$$

$$\mathbf{u}|_{\Gamma} = \mathbf{0}, \tag{3}$$

$$\mathbf{u}(x,0) = \mathbf{u}^0(x), \tag{4}$$

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where $\mathbf{u}(\cdot, t)$ is a vector field in some domain Ω in $\mathbb{R}^K$ with boundary Γ and operator $-\Delta$ acts on the components of $\mathbf{u}$. This problem arises in the study of the general Navier–Stokes equations; see [10–12,17,19,26], for example. An approximate solution must accommodate the incompressibility requirement (2) somehow; thus, some special adjustment to the usual finite element approximations must be made.

Here, we apply the cell discretization algorithm, a nonconforming extension of the finite element method due to Greenstadt [13,14] and Raviart and Thomas [20]. See also [9]. The method is similar to the mortar method of Bernardi, Maday et al. [4–6,18] and yields similar error estimates.

A domain is partitioned into *cells* and solutions are approximated by linear combinations of basis functions on each cell. Another set of basis functions defined on the interfaces between cells is used to achieve weak continuity over the entire domain by requiring that the difference of the traces of approximations on the common boundaries of adjacent cells be orthogonal to increasing numbers of the interface basis. These requirements, called *moment collocations*, are expressed as a set of linear constraints on the coefficients that are used to define the approximation to the solution on each cell. Here we impose additional linear constraints enforcing a weak approximation to the solenoidal requirement to obtain an approximate solution to the Stokes equations. For the stationary Stokes equations, convergence of approximations to both the velocity $\mathbf{u}$ and the pressure Π is shown in [22,25]. Convergence of approximations to parabolic equations using the cell discretization algorithm is established in [24]. The methods in these two papers provide the framework for the results shown here.

In Section 1 we obtain general error estimates that prove convergence of approximations to the velocity $\mathbf{u}$ and the pressure Π.

In Section 2 we describe an implementation of the method for problems with domains consisting of unions of triangles and parallelograms using polynomial bases. We can produce the continuous approximations of the h–p finite element method [1–3,7,16] that also satisfy the solenoidal condition exactly. We give specific error estimates in terms of the diameter h of a cell and the degree p of an approximation on the cell and in Section 3 describe experiments that substantiate the theory.

1. Theoretical results

The setting for the cell discretization algorithm is given in detail in [23]. The method is extended to the stationary Stokes equations in [22,25]. We give a summary here.

We assume that bounded domain Ω in $\mathbb{R}^K$ has a Lipschitz boundary that is piecewise C^1(referred to as an *LPC^1 domain*).

The Hilbert spaces we use are the following:

Let $(\cdot,\cdot)_0$ denote the $L_2(\Omega)$ inner product, with norm denoted $\|\cdot\|_0$.

Let $\mathbf{L}_2(\Omega) \equiv L_2(\Omega) \times L_2(\Omega) \times \cdots \times L_2(\Omega)$, the Cartesian product of K copies of $L_2(\Omega)$. In $\mathbf{L}_2(\Omega)$, bold face symbols denote vector fields; e.g. $\mathbf{u} = (u^1, \dots, u^K)$.

We define $(\mathbf{u}, \mathbf{v})_0 = (u^1, v^1)_0 + \cdots + (u^K, v^K)_0$. The associated norm is denoted $\|\mathbf{u}\|_0^2 = (\mathbf{u}, \mathbf{u})_0$.

$H^1(\Omega) \equiv \{u : \Omega \to \mathbb{R}\colon u \in L_2(\Omega); D_i u \in L_2(\Omega)$ for $i = 1, \dots, K\}$ where partial derivatives $D_i u$ are distribution derivatives with respect to x_i. This is a Hilbert space when endowed with inner product

$$(u, v)_1 \equiv \sum_{i=1}^{K} (D_i u, D_i v)_0 + (u, v)_0.$$

The $H^1(\Omega)$ norm is denoted $\|\cdot\|_1$.

For vector fields $\mathbf{u}$, we define $\mathbf{H}^1(\Omega)=H^1(\Omega)\times\cdots\times H^1(\Omega)$. The inner product here is

$$(\mathbf{u},\mathbf{v})_1=(u^1,v^1)_1+\cdots+(u^K,v^K)_1.$$

The norm is $\|\mathbf{u}\|_1^2=(\mathbf{u},\mathbf{u})_1$.

We use $C_0^\infty(\Omega)$ to represent the set of infinitely differentiable functions with compact support in Ω and let $H_0^1(\Omega)$ denote its closure in the H^1-norm. Bold-face $\mathbf{H}_0^1(\Omega)$ represents the subspace of $\mathbf{H}^1(\Omega)$ consisting of K copies of $H_0^1(\Omega)$.

Following Greenstadt's cell discretization method, we allow domain Ω to be partitioned into N LPC^1 subdomains $\Omega_1,\dots,\Omega_N$, with $\Omega_i\cap\Omega_j=\emptyset$ if $i\neq j$ and $\bar{\Omega}=\bigcup_{k=1}^N\bar{\Omega}_k$. The Ω_k are called *cells*. Let $\Omega_0\equiv\mathbb{R}^K\setminus\bar{\Omega}$.

Let $(\cdot,\cdot)_{0,i}$ and $(\cdot,\cdot)_{1,i}$ denote the $L_2(\Omega_i)$ and $H^1(\Omega_i)$ inner products on cell Ω_i. The norms are represented by $\|\cdot\|_{0,i}$ and $\|\cdot\|_{1,i}$, respectively.

We define $H=\{u\in L_2(\Omega)\colon u|_{\Omega_k}\in H^1(\Omega_k); k=1,\dots,N\}$. H is a Hilbert space with inner product $(u,v)_H=\sum_{k=1}^N(u,v)_{1,k}$.

When extended to a vector field $\mathbf{u}$, the notation is $\mathbf{H}$, with inner product $(\mathbf{u},\mathbf{v})_H=\sum_{i=1}^K(u^i,v^i)_H$. The norm is denoted by $\|\cdot\|_H$. For $\mathbf{u}\in\mathbf{H}$, we let $\nabla\cdot\mathbf{u}$ denote the scalar function defined a.e. in Ω by defining it on each cell as $\nabla\cdot\mathbf{u}|_{\Omega_k}=\sum_{i=1}^K D_i(u^i|_{\Omega_k})$.

Let $\Gamma_{ij}=\bar{\Omega}_i\cap\bar{\Omega}_j$. ($\Gamma_{ij}$ may have several C^1 components; for simplicity we assume that there is only one; see [23].) We denote by γ_{ij} the trace operator restricting u defined on Ω_i to its values on Γ_{ij}. The inner product for $L_2(\Gamma_{ij})$ is denoted by $\langle\cdot,\cdot\rangle_{ij}$, with norm represented by $\|\cdot\|_{ij}$. There are constants C_{ij} such that for any $w\in H$, $\|\gamma_{ij}(w)\|_{ij}\leqslant C_{ij}\|w\|_{1,i}$.

For each Γ_{ij}, let $\{\omega_k^{ij}\}_{k=1}^\infty$ be a set of functions in $H^{1/2}(\Gamma_{ij})$ that are a basis for $L_2(\Gamma_{ij})$. For any n, suppose that F_n^{ij} is the linear span of $\{\omega_k^{ij}\}_{k=1}^n$. For any $h\in L_2(\Gamma_{ij})$, let $S_n^{ij}(h)$ denote the $L_2(\Gamma_{ij})$ orthogonal projection onto F_n^{ij}, so that $T_n^{ij}(h)\equiv h-S_n^{ij}(h)$ satisfies $\|T_n^{ij}(h)\|_{ij}=\inf\{\|h-\phi_n^{ij}\|_{ij}\colon \phi_n^{ij}\in F_n^{ij}\}$.

We suppose that for any $h\in L_2(\Gamma_{ij})$ and for any $\varepsilon>0$, there is some $N(h,\varepsilon)$ such that $n>N(h,\varepsilon)$ $\Rightarrow\|\mathcal{T}_n^{ij}(h)\|_{ij}<\varepsilon$. Since $\{\omega_k^{ij}\}_{k=1}^\infty$ is a basis, there is some q such that $\langle 1,\omega_q^{ij}\rangle_{ij}\neq 0$.

For $u\in H$, we define the kth *moment* of $u|_{\Omega_i}$ on Γ_{ij} to be $M_k^{ij}(u)\equiv\langle\gamma_{ij}(u),\omega_k^{ij}\rangle_{ij}$. To make an approximation weakly continuous on Ω, we require some of the moments of an approximation u to be equal on interfaces Γ_{ij} in the following way.

Let N_I be the number of interfaces Γ_{ij}. $[n]$ denotes a multi-index, an N_I-vector of nonnegative integers $(\dots,n_{ij},\dots)$, with integer n_{ij} associated with interface Γ_{ij}. Let $G[n]\equiv\{u\in H\colon$ for any $ij, j\neq 0$, $ij=1,\dots,N_I$ and for any $k\leqslant n_{ij}$, we have $M_k^{ij}(u)=M_k^{ji}(u)\}$; this is the set of functions u in H such that the difference of the traces from either side of any internal interface $\Gamma_{ij},\gamma_{ij}(u)-\gamma_{ji}(u)$, is $L_2(\Gamma_{ij})$-orthogonal to ω_k^{ij}, $k=1,\dots,n_{ij}$. This gives a notion of weak continuity across interfaces called *moment collocation*.

Let $G_0[n]=\{u\in G[n]\colon$ for any i and for any $k\leqslant n_{i0}, M_k^{i0}(u)=0\}$; this is the set of functions in $G[n]$ that are weakly 0 on the external interfaces Γ_{i0} making up Γ.

We define a partial order for such multi-indices; we say $[n']\geqslant[n]\Leftrightarrow$ for any $ij, n'_{ij}\geqslant n_{ij}$. If $[n^k]$ is a sequence of multi-indices, $k=1,2,\dots$, we say that $[n^k]\to[\infty]$ if $[n^k]\leqslant[n^{k+1}]$ and $\inf\{n_{ij}^k\}\to\infty$ as $k\to\infty$.

When $\mathbf{u}$ is a vector field, we denote the appropriate extension of our definitions by $\mathbf{G}[\mathbf{n}]$ and $\mathbf{G}_0[\mathbf{n}]$.

We need a space for our approximations. For any Ω_k, let $\{B_i^k,\ i=1,\dots\}$ be a basis for $H^1(\Omega_k)$. For any m and any $v\in H$, let $S_m^k(v)$ denote the $H^1(\Omega_k)$ orthogonal projection onto F_m^k, the linear span of $\{B_i^k\}_{i=1}^m$. Then S_m is the operator defined for $v\in H$ by its expression on each cell given by $S_m(v)|_{\Omega_k}\equiv S_m^k(v)$. For v in H, we define the residual $T_m(v)\equiv v-S_m(v)$. Then

$$\|T_m(v)\|_1^2=\inf\left\{\sum_{i=1}^N\|v-\phi_m^i\|_{1,i}^2\colon\ \phi_m^i\in F_m^k\right\}.$$

For any $v\in\mathbf{H}$ and for any $\varepsilon>0$, there is some $N(v,\varepsilon)$ such that $n>N(v,\varepsilon)\Rightarrow\|T_m(v)\|_1<\varepsilon$.

For a vector field $\mathbf{v}=(v^1,\dots,v^k)$, we let $\mathbf{Q}_m(\mathbf{v})$ denote the operator defined on each component v^q to be $T_m(v^q)$.

The space of solenoidal vector fields is $\mathbf{V}\equiv\{\mathbf{u}\in\mathbf{H}^1(\Omega)\colon\ \nabla\cdot\mathbf{u}=0\}$. This implies that for $\mathbf{u}\in\mathbf{V}$ and any scalar function $\rho\in H^1(\Omega)$, $(\nabla\cdot\mathbf{u},\rho)_0=0$. We adapt this requirement to produce a weak solenoidal condition as follows:

Set $\mathbf{V}[\mathbf{r}]=\{\mathbf{u}\in\mathbf{H}:(\nabla\cdot\mathbf{u},B_i^k)_{0,k}=0$ for $k=1,\dots,N$ and $i\leqslant r\}$. We have the inclusions $r'>r\Rightarrow\mathbf{V}[\mathbf{r}']\subset\mathbf{V}[\mathbf{r}]$.

Set $\mathbf{G}_0[\mathbf{n}][\mathbf{r}]\equiv\mathbf{G}_0[\mathbf{n}]\cap\mathbf{V}[\mathbf{r}]$. We have the inclusions $[n']\geqslant[n]$ and $r'\geqslant r\Rightarrow\mathbf{G}_0[\mathbf{n}'][\mathbf{r}']\subset\mathbf{G}_0[\mathbf{n}][\mathbf{r}]$.

Finally, we define our approximation space:

N is the number of cells in the domain decomposition; let $[m]$ be an N-dimensional multi-index indicating the number of basis functions used in the approximation on each cell.

Let $\mathbf{H}[\mathbf{m}]=\{\mathbf{u}\in\mathbf{H}:u^q|_{\Omega_k}=\sum_{j=1}^{m_k}b_j^{kq}B_j^k\}$.

Let $\mathbf{G}_0[\mathbf{m}][\mathbf{n}][\mathbf{r}]\equiv\mathbf{H}[\mathbf{m}]\cap\mathbf{G}_0[\mathbf{n}][\mathbf{r}]$. This is a finite dimensional space; the moment collocation requirements are

$$(\langle\gamma_{ij}(u^q),\omega_p^{ij}\rangle_{ij}-\langle\gamma_{ji}(u^q),\omega_p^{ij}\rangle_{ij})=0,\quad q=1,\dots,K;\quad p=1,\dots,n_{ij},$$

and

$$\langle\gamma_{i0}(u^q),\omega_p^{i0}\rangle_{i0}=0,\quad q=1,\dots,K;\quad p=1,\dots,n_{i0}. \tag{5}$$

These requirements produce linear constraints among the b_i^{kq} for each $q,q=1,\dots,K$.

The weak solenoidal requirement is

$$\sum_{q=1}^K(B_i^k,D_qu^q)_{0,i}=0,\quad i=1,\dots,r;\quad k=1,\dots,N. \tag{6}$$

This requirement produces more linear relations among the b_i^{kq}.

We also use space $\mathbf{G}_0[\mathbf{m}][\mathbf{n}]\equiv\mathbf{H}[\mathbf{m}]\cap\mathbf{G}_0[\mathbf{n}]$ in our discussion of approximations to the pressure Π.

A crucial lemma concerns the following projections: we define $\mathbf{P}_{mn}^r$ to be the $\mathbf{H}$-orthogonal projection operator mapping $\mathbf{G}_0[\mathbf{n}][\mathbf{r}]$ onto $\mathbf{G}_0[\mathbf{m}][\mathbf{n}][\mathbf{r}]$.

Lemma 1.1. *For any* $[n]$ *and* $[r]$, *there is a parameter* $K(n,r)>0$ *depending on* $[n],[r]$ *and the geometry of the cell structure partitioning the domain such that for any* $\mathbf{v}\in\mathbf{G}_0[\mathbf{n}][\mathbf{r}]$,

$$\|\mathbf{v}-\mathbf{P}_{mn}^r\mathbf{v}\|_H\leqslant K(n,r)\|\mathbf{Q}_m(\mathbf{v})\|_H.$$

Thus

$$\inf\{\|\mathbf{v}-\mathbf{w}\|_H\colon \mathbf{w}\in \mathbf{G}_0[\mathbf{m}][\mathbf{n}][r]\} \leqslant K(n,r)\inf\{\|\mathbf{v}-\mathbf{w}\|_H\colon \mathbf{w}\in \mathbf{H}[\mathbf{m}]\}.$$

Let $C \equiv \sup\{C_{ij}\}$.

(a) *Define* c_i *to be the maximum number of collocations employed on all interfaces* Γ_{ij} *of cell* Ω_i. *Let* $m_c = \sup_i\{c_i\}$. *Suppose* $\|\omega_p^{ij}\|_{ij}^2 \leqslant W$ *for all* Γ_{ij}. *Then*

$$K(n,r) \leqslant (1+(1/\mu)[2C^2 W m_c + Kr])^{1/2}.$$

(b) *If the collocation weight functions* ω_k^{ij} *are* $L_2(\Gamma_{ij})$ *torthonormal, and* n_f *is the maximum number of* C^1 *faces of any cell, then*

$$K(n,r) \leqslant (1+(1/\mu)[2n_f C^2 + Kr])^{1/2}.$$

Parameter μ *is the least eigenvalue of a positive-definite matrix that depends on the bases and cell structure and* $[n]$ *and* r. *We describe this matrix and list the properties of* μ *in [25].* $1/\mu$ *is nonincreasing as* $[m]\to[\infty]$. *Thus it follows that* $\lim_{[m]\to[\infty]}\|\mathbf{v}-\mathbf{P}_{mn}^r\mathbf{v}\|_H = 0$ *for any* $\mathbf{v}\in \mathbf{G}_0[\mathbf{n}][\mathbf{r}]$.

Values for $1/\mu$ for a polynomial implementation are given in the next section.

The estimate in (b) is used in Section 2 where we describe a polynomial implementation of the algorithm for triangular and parallelogram cells Ω_k. The bases we use for such cells in $\mathbb{R}^2$ are $L_2(\Omega_k)$-orthonormal, and thus provide appropriate $L_2(\Gamma_{ij})$-orthonormal collocation weight functions for the parallelogram or triangular interfaces between parallelepiped or tetrahedral cells in $\mathbb{R}^3$ so that the estimate in (b) can be applied in this situation as well.

We generate a basis for $\mathbf{G}_0[\mathbf{m}][\mathbf{n}][\mathbf{r}]$ that satisfies both the collocation constraints and the weak solenoidal requirement. The method generalizes the following algorithm for domains in $\mathbb{R}^2$.

The coefficients $\{b_i^{kq}\}$ for the representation on each of the N cells can be concatenated to form vector $\mathbf{b}^{\mathrm{T}} \equiv (\mathbf{b}_1^{\mathrm{T}}, \mathbf{b}_2^{\mathrm{T}})$, where $\mathbf{b}_i^{\mathrm{T}}$ denotes $(b_1^{1i}, b_2^{1i}, \ldots, b_1^{2i}, b_2^{2i}, \ldots, b_1^{ki} b_2^{ki} \ldots)$. The linear moment collocation requirements (5) are expressed as $\mathbf{M}_i\mathbf{b}_i = \mathbf{0}$. The matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are the same; they are $n'\times m'$ matrices where $n' = \sum n_{ij}$ and $m' = \sum_{i=1}^N m_i$; we will have $m' > n'$. In [23] it is shown that the rows of $\mathbf{M}_i$ are independent if $[m]$ is sufficiently large and we suppose that this is the case.

The linear weak solenoidal requirement (6) is represented as $(\mathbf{S}_1|\mathbf{S}_2)\mathbf{b} = \mathbf{0}$, where matrices $\mathbf{S}_q$ have entries of form $(B_i^k, D_q B_j^k)_{0,i}$, $q = 1,2$, $k = 1,\ldots,N$, $i = 1,\ldots,r$ and, for each $k, j = 1,\ldots,m_k$. Thus, for our two-dimensional case, $(\mathbf{S}_1|\mathbf{S}_2)$ is of size $(Nr)\times(2m')$. In [22], we show that there is also some sufficiently large $[m']$ so that $(\mathbf{S}_1|\mathbf{S}_2)$ is of full rank and we suppose that this is so.

Thus the coefficients $\{b_i^{kq}\}$ must satisfy the conditions

$$\mathbf{M}_1\mathbf{b}_1 = \mathbf{0}, \tag{7}$$

$$\mathbf{M}_2\mathbf{b}_2 = \mathbf{0}, \tag{8}$$

and

$$\mathbf{S}_1\mathbf{b}_1 + \mathbf{S}_2\mathbf{b}_2 = \mathbf{0}. \tag{9}$$

If we define $\mathscr{M}$ to be the matrix

$$\begin{pmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \\ \mathbf{S}_1 & \mathbf{S}_2 \end{pmatrix},$$

these requirements are succinctly expressed as $\mathscr{M}\mathbf{b}=\mathbf{0}$ or $\mathbf{b}^{\mathrm{T}}\mathscr{M}^{\mathrm{T}}=\mathbf{0}^{\mathrm{T}}$; the set of acceptable b_i^{kq} is the null space of $\mathscr{M}$. $\mathscr{M}$ is an n by m matrix, where $n=2n'+Nr$ and $m=2m'$. If there is any dependency between the rows of $\mathscr{M}$, it is shown in [22] that we can delete suitable rows so that, without loss of generality, we can assume that $\mathscr{M}$ is of full rank.

To construct a basis that satisfies (7)–(9), obtain the 'QR' factorization of $\mathscr{M}^{\mathrm{T}}$, so $\mathscr{M}^{\mathrm{T}}=(\mathbf{Q}''|\mathbf{Q}')\binom{\mathbf{R}}{\mathbf{O}}$, where $\mathbf{R}$ is square upper-triangular and invertible and $Q\equiv(\mathbf{Q}''|\mathbf{Q}')$ is orthogonal. Since we are looking for $\mathbf{b}$ such that $\mathbf{b}^{\mathrm{T}}\mathscr{M}^{\mathrm{T}}=\mathbf{0}^{\mathrm{T}}$, an easy argument shows that the columns of $\mathbf{Q}'$, the last $m-n$ columns of Q, are an orthonormal basis for the null space of $\mathscr{M}$.

Let $p\equiv m-n=2m'-2n'-Nr$ and suppose that the p columns of $\mathbf{Q}'$ are

$$(q_{11},\ldots,q_{m1})^{\mathrm{T}},(q_{12},\ldots,q_{m2})^{\mathrm{T}},\ldots,(q_{1p},\ldots,q_{mp})^{\mathrm{T}}.$$

Let bold face $\mathbf{B}_i^{k1}\equiv(B_i^k,0)$ defined on Ω by assuming it is zero outside Ω_k; likewise, $\mathbf{B}_i^{k2}$ is the pair $(0,B_i^k)$. We enumerate these $\{\mathbf{B}_i^{kq}\}$ as

$$(\mathbf{B}_1^{11},\mathbf{B}_2^{11},..,\mathbf{B}_{m_1}^{11},\mathbf{B}_1^{21},\mathbf{B}_2^{21},..,\mathbf{B}_{m_2}^{21},\ldots;\mathbf{B}_1^{12},\mathbf{B}_2^{12},..,\mathbf{B}_{m_1}^{12},\mathbf{B}_2^{22},\ldots);$$

there are m such $\mathbf{B}_i^{kq}$. Denote the $\mathbf{B}_i^{kq}$ with this enumeration as $\{\Phi^1,\Phi^2,\ldots,\Phi^m\}$ and form $\mathscr{B}_i\equiv\sum_{j=1}^m q_{ji}\Phi^j$. Then $\{\mathscr{B}_i\}$ is a basis for $\mathbf{G}_0[\mathbf{m}][\mathbf{n}][\mathbf{r}]$. Any approximation of form $\mathbf{u}_{mn}^r=\sum_{i=1}^p y_i\mathscr{B}_i$ can be expressed in terms of the original basis represented by $\{\Phi^1,\Phi^2,\ldots,\Phi^m\}$ in the following way:

$$\sum_{i=1}^p y_i\mathscr{B}_i=\sum_{i=1}^p y_i\sum_{j=1}^m q_{ji}\Phi^j=\sum_{j=1}^m\sum_{i=1}^p y_iq_{ji}\Phi^j.$$

Thus, the coefficients of Φ^j are $\phi_j\equiv\sum_{i=1}^p y_iq_{ji}$, which are the components of vector $\phi=\mathbf{Q}'\mathbf{y}$, where $\mathbf{y}$ is the column matrix of the y_i.

For simplicity of exposition, we assume that $\{\mathscr{B}_i\}$ is an $L_2(\Omega)$ orthonormal set. Since the columns of $\mathbf{Q}'$ are orthonormal, a straightforward argument shows that it is sufficient that the original basis functions $\{B_i^k\}$ defined on Ω_k be an $L_2(\Omega_k)$ orthonormal set. We use orthonormal basis functions in our implementation of the method for domains in $\mathbb{R}^2$ described in Section 2. These arguments extend to $\mathbb{R}^3$ if we use parallelepiped or tetrahedral cells; products of Legendre polynomials give L_2-orthonormal basis functions for parallelepipeds and an L_2-orthonormal polynomial basis for tetrahedra has been constructed by Hui [15].

We extend these definitions to an appropriate setting for time-dependent problems.

Define $\Omega^T\equiv\Omega\times[0,T]$ and let $\mathbf{H}_0^{1,T}(\Omega)\equiv C([0,T]:\mathbf{H}_0^1(\Omega))$.

Approximations are in $\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}]\equiv C([0,T];\mathbf{G}_0[\mathbf{m}][\mathbf{n}][\mathbf{r}])$. Members $\mathbf{v}$ of $\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}]$ are represented as $\mathbf{v}\equiv\sum_{i=1}^p y_i(t)\mathscr{B}_i(x)$.

We use space $\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}]\equiv C\,([0,T];\mathbf{G}_0[\mathbf{m}][\mathbf{n}])$ in our discussion of pressure.

We define bilinear form $a(u,v)_i$ acting on $H^1(\Omega_i)\times H^1(\Omega_i)$ to be $\int_{\Omega_i}\sum_{j=1}^k D_juD_jv\,\mathrm{d}x$ and $a(u,v)\equiv\sum_{i=1}^N a(u,v)_i$. It is shown in [8] that if $[n]$ is sufficiently large so that, for each Γ_{ij}, there is some ω_k^{ij} such that $\langle\omega_k^{ij},1\rangle_{ij}\neq 0$ with $k\leqslant n_{ij}$, then $a(\cdot,\cdot)$ is coercive over $G_0[n]$.

For $\mathbf{u},\mathbf{v}\in\mathbf{H}$, let $\mathbf{a}(\mathbf{u},\mathbf{v})=\sum_{q=1}^K a(u^q,v^q)$ where $\mathbf{u}=(u^1,\ldots,u^q,\ldots,u^K)$ and $\mathbf{v}$ is similarly represented. Due to the coercivity of $a(\cdot,\cdot)$, there is a positive constant c such that $\mathbf{a}(\mathbf{u},\mathbf{u})\geqslant c\|\mathbf{u}\|_H^2$ for any $\mathbf{u}\in\mathbf{G}_0[\mathbf{n}]$.

Let $\mathbf{f}:\Omega^T\to\mathbb{R}^K$ have components f^q in $C([0,T];L_2(\Omega))$. Initial value $\mathbf{u}^0$ is in $\mathbf{H}_0^1(\Omega)\cap\mathbf{V}$.

A weak variational form of the Stokes problem is conventionally considered ([26], p. 253 ff.) For our purposes, we express it as the task of finding some $\mathbf{u}$ in $C([0,T];\mathbf{H}_0^1(\Omega)\cap\mathbf{V})\cap C^1([0,T];\mathbf{L}_2(\Omega))$ such that for all $\mathbf{v}\in C([0,[T];\mathbf{H}_0^1(\Omega)\cap\mathbf{V})$,

$$(\mathbf{u}',\mathbf{v})_0+\mathbf{a}(\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v})_0, \tag{10}$$

$$\mathbf{u}(\cdot,t)=\mathbf{u}^0,$$

where $\mathbf{u}'$ is the vector of t-derivatives of the components of $\mathbf{u}$.

Our approximate solution is obtained by solving (10) over the space $\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}]$. Thus, we must find some $\mathbf{u}_{mn}^r=\sum_{i=1}^p y_i(t)\mathscr{B}_i\in\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}]$ such that

$$(\mathbf{u}_{mn}^{r}{}',\mathscr{B}_k)_0+\mathbf{a}(\mathbf{u}_{mn}^r,\mathscr{B}_k)=(\mathbf{f},\mathscr{B}_k)_0=\left(\mathbf{f},\sum_{j=1}^m q_{jk}\Phi^j\right)_0 \quad \text{for } k=1,\ldots,p$$

and

$$\mathbf{u}_{mn}^r|_{t=0}=\mathbf{u}_{mn}^{r0}, \tag{11}$$

where we approximate the initial data by the L_2 orthogonal projection of $\mathbf{u}^0$ into $\mathbf{G}_0[\mathbf{m}][\mathbf{n}]$: $\mathbf{u}_{mn}^{r0}(x,0)\equiv\sum_{i=1}^p d_i\mathscr{B}_i$ where $d_i\equiv(\mathbf{u}^0,\mathscr{B}_i)_0=(\mathbf{u}^0,\sum_{j=1}^m q_{ji}\Phi^j)_0$.

Substituting for $\mathbf{u}_{mn}^r=\sum_{i=1}^p y_i(t)\mathscr{B}_i$ in Eq. (8), we get

$$\sum_{i=1}^p y_i(t)'(\mathscr{B}_i,\mathscr{B}_k)_0+\sum_{i=1}^p y_i(t)\mathbf{a}(\mathscr{B}_i,\mathscr{B}_k)=(\mathbf{f},\mathscr{B}_k)_0,\quad k=1,\ldots,p,$$

and

$$y_i(0)=d_i. \tag{12}$$

Now we are assuming that $(\mathscr{B}_i,\mathscr{B}_k)_0=\delta_i^k$, so, writing in vector notation, Eq. (12) becomes

$$\mathbf{y}'+(\mathbf{a}(\mathscr{B}_i,\mathscr{B}_k))\mathbf{y}=((\mathbf{f},\mathscr{B}_k)_0)^{\mathrm{T}}$$

and

$$\mathbf{y}(0)=\mathbf{d}. \tag{13}$$

Entries in the matrix are

$$\mathbf{a}(\mathscr{B}_i,\mathscr{B}_j)=\mathbf{a}\left(\sum_k^m q_{ki}\Phi^k,\sum_q^m q_{qj}\Phi^q\right)=\sum_{k,q}^m q_{ki}\mathbf{a}(\Phi^k,\Phi^q)q_{qj}.$$

These are the entries of $\mathbf{Q}'^{\mathrm{T}}\mathbf{C}\mathbf{Q}'$, where (in $\mathbb{R}^K$) $\mathbf{C}$ is a matrix of K identical diagonal blocks; each block consists of N diagonal positive-semi-definite blocks of size $m_k\times m_k$, one for each cell Ω_k. The entries in these blocks are $a(B_i^k,B_j^k)_k$ expressed in terms of the original basis for $H^1(\Omega_k)$. However, with the assumption that, for each Γ_{ij}, there is some ω_k^{ij} such that $\langle\omega_k^{ij},1\rangle_{ij}\neq 0$ with $k\leqslant n_{ij}$, the resulting coercivity of $\mathbf{a}(\cdot,\cdot)$ implies that $\mathbf{Q}'^{\mathrm{T}}\mathbf{C}\mathbf{Q}'$ is symmetric and positive definite [8]. Then system (13) of linear ordinary differential equations is solved using classical methods.

For a two-dimensional problem, the size of the matrix $\mathbf{Q}'^{\mathrm{T}}\mathbf{C}\mathbf{Q}'$ is $p\times p$, where $p=2m'-2n'-Nr$. For the implementation in Section 2 using a polynomial basis, for a moderate problem with 10 rectangular cells, a tenth degree polynomial basis, 7 collocations on each Γ_{ij} and the solenoidal requirement enforced exactly, p is about 400, small enough to enable us to obtain the necessary eigenvalues and eigenvectors to obtain a solution. The p degrees of freedom in the system pertain to the approximation of the solution given $\mathbf{f}$ and $\mathbf{u}^0$; we have eliminated any concern with weak continuity between interfaces of cells and the solenoidal requirement.

Any $\mathbf{v}\in\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}]$ is of form $\sum_{k=1}^{p} z_k(t)\mathscr{B}_k$, and if we multiply Eq. (12) by $z_k(t)$ for each k and sum over k, we get

$$(\mathbf{u}^{r\,\prime}_{mn},\mathbf{v})_0+\mathbf{a}(\mathbf{u}^r_{mn},\mathbf{v})=(\mathbf{f},\mathbf{v})_0. \tag{14}$$

Again using the two-dimensional case as representative, the time-dependent coefficients $b_i^{kq}(t)$ that express the approximate solution in terms of the original basis are obtained from the relation $\mathbf{b}=\mathbf{Q}'\mathbf{y}(t)$. The coefficients $e_i^{kq}(t)$ that give $\mathbf{v}\in\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}]$ in terms of the original basis are likewise given by $\mathbf{e}=\mathbf{Q}'\mathbf{z}(t)$; the collocations and weak solenoidal condition are satisfied by requiring that both $\mathscr{M}\mathbf{b}=\mathbf{0}$ and $\mathscr{M}\mathbf{e}=\mathbf{0}$. If $\mathbf{C}$ is the (symmetric) matrix $(\mathbf{a}(\Phi^k,\Phi^q))$ described above consisting of diagonal blocks, (14) is expressed as the equation

$$(\mathbf{b}')^{\mathrm{T}}\mathbf{e}+\mathbf{b}^{\mathrm{T}}\mathbf{C}\mathbf{e}=((\mathbf{f},\Phi^j)_0)^{\mathrm{T}}\mathbf{e} \tag{15}$$

with $\mathbf{b}(0)=((\mathbf{u}^0,\Phi^j)_0)$. Thus vector $\mathbf{b}'+\mathbf{C}\mathbf{b}-(\mathbf{f},\Phi^j)_0$ is orthogonal to the null space of M. This observation enables us to obtain an approximation to the pressure Π.

We need Green's formula in the proofs that follow. With $D_{\mathbf{n}}\mathbf{u}$ denoting vector of outward normal derivatives of the components of $\mathbf{u}$, Green's formula is

$$(-\Delta\mathbf{u},\mathbf{v})=\mathbf{a}(\mathbf{u},\mathbf{v})-\langle D_{\mathbf{n}}\mathbf{u},\gamma(\mathbf{v})\rangle_{\Gamma'},$$

where $\langle\cdot,\cdot\rangle$ denotes the vector $L_2(\Gamma')$ inner product. This is valid for LPC^1 domains Ω' with boundary Γ' for $\mathbf{u}$ in $\mathbf{H}^2(\Omega')$ and $\mathbf{v}$ in $\mathbf{H}^1(\Omega')$. In particular this holds for $\Omega'=\Omega$ or $\Omega'=$ any Ω_k.

With these preparations, we state the convergence result. Sufficient conditions for the required regularity for solution $\mathbf{u}(x,t)$ are discussed in [26]. Results there are proved under the assumption that Γ is of class C^2. In the statement of the theorem, the unit normal to Γ_{ij} (pointing outward with respect to cell Ω_i) is denoted $v^{\cdot}_{ij}\equiv(v^1_{ij},\dots,v^K_{i})$.

Theorem 1.2. *Suppose that* $\mathbf{u}^0\in\mathbf{H}_0^1(\Omega)\cap\mathbf{V}$ *and* $\mathbf{f}\in C([0,T];\mathbf{L}_2(\Omega))$. *Assume that our solution* $\langle\mathbf{u},\Pi\rangle$ *of* (1),…,(4) *satisfies the following conditions*: $\mathbf{u}\in C([0,T];\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega))$, $\partial\mathbf{u}/\partial t\in C([0,T];\mathbf{H}_0^1(\Omega))$, $\mathbf{u}(x,0)=\mathbf{u}^0(x)$ *and* $\Pi\in C([0,T];H^1(\Omega))$.

Suppose $\mathbf{u}^r_{mn}$ *is the approximation obtained by solving* (13). *Then, for any* $\tau\in[0,T]$,

$$\begin{aligned}
&\|\mathbf{u}(\tau)-\mathbf{u}^r_{mn}(\tau)\|_0^2+c\int_0^\tau\|\mathbf{u}-\mathbf{u}^r_{mn}\|_H^2(t)\,\mathrm{d}t\\
&\quad\leqslant 2K(n,r)^2(3\|\mathbf{Q}_m\mathbf{u}^0\|_H^2+2\|\mathbf{Q}_m\mathbf{u}(\tau)\|_H^2)\\
&\qquad+\frac{20}{3c}\int_0^\tau[K(n,r)^2\{\|\mathbf{Q}_m\mathbf{u}\|_H^2+\|\mathbf{Q}_m\mathbf{u}'\|_H^2\}+K\|\mathscr{T}_r(\Pi)\|_0^2]\,\mathrm{d}t\\
&\qquad+Nn_f^2\sup\{C_{ij}^2\}\frac{20}{3c}\int_0^\tau(\sup\{\|\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^{\cdot}_{ij})\|_{ij}^2\}+\sup\{\|\mathscr{T}^{ij}_{n_{ij}}(D_{\mathbf{n}_i}\mathbf{u})\|_{ij}^2\})\,\mathrm{d}t,
\end{aligned}$$

where n_f is the maximum number of C^1 faces Γ_{ij} on any cell, $K(n,r)$ is the parameter of Lemma 1.1. *and $\gamma_{ij}(\Pi)v_{ij}$ is the scalar trace of pressure Π on Γ_{ij} times the unit normal vector to Γ_{ij}.*

We convert the problem and establish a number of estimates before returning to a formal proof. The basic argument concerns

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}-\mathbf{u}^r_{mn}\|_0^2 = (\mathbf{u}'-\mathbf{u}^r_{mn}{}',\mathbf{u}-\mathbf{u}^r_{mn})_0$$
$$= (\mathbf{u}'-\mathbf{u}^r_{mn}{}',\mathbf{u}-\mathbf{P}^r_{mn}\mathbf{u})_0 + (\mathbf{u}'-\mathbf{u}^r_{mn}{}',\mathbf{P}^r_{mn}\mathbf{u}-\mathbf{u}^r_{mn})_0. \tag{16}$$

The solution satisfies $(\mathbf{u}'-\Delta\mathbf{u}+\nabla\Pi-\mathbf{f},\mathbf{v})_0=0$, or

$$(\mathbf{u}',\mathbf{v})_0 = -(-\Delta\mathbf{u},\mathbf{v})_0-(\nabla\Pi,\mathbf{v})_0+(\mathbf{f},\mathbf{v})_0 \quad \text{for } \textit{any } \mathbf{v}\in\mathbf{H}.$$

As in [25], using Green's formula on each cell, we get, for $\mathbf{v}\in\mathbf{H}$,

$$(-\Delta\mathbf{u},\mathbf{v})_0 = \mathbf{a}(\mathbf{u},\mathbf{v}) - \sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\mathbf{v})-\gamma_{ji}(\mathbf{v})\rangle_{ij} - \sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_{i0}}\mathbf{u},\gamma_{i0}(\mathbf{v})\rangle_{i0}.$$

Thus,

$$(\mathbf{u}',\mathbf{v})_0 = -\mathbf{a}(\mathbf{u},\mathbf{v}) + \sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\mathbf{v})-\gamma_{ji}(\mathbf{v})\rangle_{ij}$$
$$+\sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_{i0}}\mathbf{u},\gamma_{i0}(\mathbf{v})\rangle_{i0} - (\nabla\Pi,\mathbf{v})_0 + (\mathbf{f},\mathbf{v})_0. \tag{17}$$

Recall that from (14), for any $\mathbf{v}\in\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}]$,

$$(\mathbf{u}^r_{mn}{}',\mathbf{v})_0 + \mathbf{a}(\mathbf{u}^r_{mn},\mathbf{v}) = (\mathbf{f},\mathbf{v})_0.$$

We use this fact and let $\mathbf{v}=\delta\equiv\mathbf{P}^r_{mn}\mathbf{u}-\mathbf{u}^r_{mn}\in\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}]$. Then

$$(\mathbf{u}'-\mathbf{u}^r_{mn}{}',\mathbf{P}^r_{mn}\mathbf{u}-\mathbf{u}^r_{mn})_0 = (\mathbf{u}',\delta)_0-(\mathbf{u}^r_{mn}{}',\delta)_0$$
$$= -\mathbf{a}(\mathbf{u},\delta) + \sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\delta)-\gamma_{ji}(\delta)\rangle_{ij} + \sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_{i0}}\mathbf{u},\gamma_{i0}(\delta)\rangle_{i0}$$
$$-(\nabla\Pi,\delta)_0 + (\mathbf{f},\delta)_0 + \mathbf{a}(\mathbf{u}^r_{mn},\delta) - (\mathbf{f},\delta)_0$$
$$= -\mathbf{a}(\mathbf{u}-\mathbf{u}^r_{mn},\delta) - (\nabla\Pi,\delta)_0 + \sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\delta)-\gamma_{ji}(\delta)\rangle_{ij}$$
$$+\sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_{i0}}\mathbf{u},\gamma_{i0}(\delta)\rangle_{i0}.$$

We write

$$-\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\delta)=-\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{P}_{mn}^r\mathbf{u}-\mathbf{u}_{mn}^r)$$

$$=\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{u}-\mathbf{u}_{mn}^r)-\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{P}_{mn}^r\mathbf{u}-\mathbf{u}).$$

Thus (16) can be expressed as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}-\mathbf{u}_{mn}^r\|_0^2=(\mathbf{u}'-\mathbf{u}_{mn}^{r}{}',\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u})_0+(\mathbf{u}'-\mathbf{u}_{mn}^{r}{}',\mathbf{P}_{mn}^r\mathbf{u}-\mathbf{u}_{mn}^r)_0$$

$$=(\mathbf{u}'-\mathbf{u}_{mn}^{r}{}',\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u})_0-\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{u}-\mathbf{u}_{mn}^r)-\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{P}_{mn}^r\mathbf{u}-\mathbf{u})$$

$$-(\nabla\Pi,\delta)_0+\sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\delta)-\gamma_{ji}(\delta)\rangle_{ij}+\sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_{i0}}\mathbf{u},\gamma_{i0}(\delta)\rangle_{i0},$$

or

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}-\mathbf{u}_{mn}^r\|_0^2+\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{u}-\mathbf{u}_{mn}^r)$$

$$=(\mathbf{u}'-\mathbf{u}_{mn}^{r}{}',\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u})_0+\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u})$$

$$-(\nabla\Pi,\delta)_0+\sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\delta)-\gamma_{ji}(\delta)\rangle_{ij}+\sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_{i0}}\mathbf{u},\gamma_{i0}(\delta)\rangle_{i0}.$$

Integrating with respect to t over $[0,\tau]$ and using inequality

$$c\|\mathbf{u}-\mathbf{u}_{mn}^r\|_H^2\leqslant\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{u}-\mathbf{u}_{mn}^r)$$

we get

$$\frac{1}{2}\|\mathbf{u}(\tau)-\mathbf{u}_{mn}^r(\tau)\|_0^2+c\int_0^\tau\|\mathbf{u}-\mathbf{u}_{mn}^r\|_H^2\,\mathrm{d}t$$

$$\leqslant\frac{1}{2}\|\mathbf{u}^0-\mathbf{u}_{mn}^{r0}\|_0^2+\int_0^\tau(\mathbf{u}'-\mathbf{u}_{mn}^{r}{}',\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u})_0\,\mathrm{d}t+\int_0^\tau\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u})\,\mathrm{d}t$$

$$-\int_0^\tau(\nabla\Pi,\delta)_0\,\mathrm{d}t+\int_0^\tau\left[\sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\delta)-\gamma_{ji}(\delta)\rangle_{ij}+\sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_{i0}}\mathbf{u},\gamma_{i0}(\delta)\rangle_{i0}\right]\mathrm{d}t. \tag{18}$$

Estimates for the right-hand side of (18) are derived in the following lemma.

Lemma 1.3. (a) $\|\mathbf{u}^0-\mathbf{u}_{mn}^{r0}\|_0\leqslant K(n,r)\|\mathbf{Q}_m\mathbf{u}^0\|_H$.

(b) $|\int_0^\tau(\mathbf{u}'-\mathbf{u}_{mn}^{r}{}',\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u})_0\,\mathrm{d}t|\leqslant\frac{1}{4}\|\mathbf{u}(\tau)-\mathbf{u}_{mn}^r(\tau)\|_0^2+\|\mathbf{u}(\tau)-\mathbf{P}_{mn}^r\mathbf{u}(\tau)\|_0^2+\frac{1}{2}\|\mathbf{u}^0-\mathbf{u}_{mn}^{r0}\|_0^2+\frac{1}{2}\|\mathbf{u}^0-\mathbf{P}_{mn}^r u^0\|_0^2+\frac{3c}{20}\int_0^\tau\|\mathbf{u}-\mathbf{u}_{mn}^r\|_H^2\,\mathrm{d}t+\frac{5}{3c}\int_0^\tau\|\mathbf{u}'-\mathbf{P}_{mn}^r\mathbf{u}'\|_0^2\,\mathrm{d}t$.

(c) $|\int_0^\tau\mathbf{a}(\mathbf{u}-\mathbf{u}_{mn}^r,\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u})\,\mathrm{d}t|\leqslant\frac{3c}{20}\int_0^\tau\|\mathbf{u}-\mathbf{u}_{mn}^r\|_H^2\,\mathrm{d}t+\frac{5}{3c}\int_0^\tau\|\mathbf{u}-\mathbf{P}_{mn}^r\mathbf{u}\|_H^2\,\mathrm{d}t$.

(d) $|\int_0^\tau(\nabla\Pi,\delta)_0\,\mathrm{d}t|\leqslant\frac{5}{3c}K\int_0^\tau\|\mathscr{T}_r(\Pi)\|_0^2\,\mathrm{d}t+\frac{5}{3c}Nn_f^2\sup\{C_{ij}^2\}\int_0^\tau\sup\{\|\mathscr{T}_{n_{ij}}^{ij}(\gamma_{ij}(\Pi)v_{ij}^{\cdot})\|_{ij}^2\}\,\mathrm{d}t+\frac{6c}{20}\int_0^\tau\|\mathbf{u}-\mathbf{u}_{mn}^r\|_H^2(t)\,\mathrm{d}t$.

(e) *If n_f denotes the maximum number of faces in any of the N cells,*

$$\left|\int_0^\tau \left[\sum_{\Gamma_{ij}} \langle D_{\mathbf{n}_{ij}}\mathbf{u}, \gamma_{ij}(\delta) - \gamma_{ji}(\delta)\rangle_{ij} + \sum_{\Gamma_{i0}} \langle D_{\mathbf{n}_{i0}}\mathbf{u}, \gamma(\delta)\rangle_{i0}\right] \mathrm{d}t\right|$$

$$\leqslant \frac{5}{3c} N n_f^2 \sup\{C_{ij}^2\} \int_0^\tau \sup\{\|\mathscr{T}^{ij}_{n_{ij}}(D_{\mathbf{n}_i}\mathbf{u})\|_{ij}^2\}\,\mathrm{d}t + \frac{3c}{20}\int_0^\tau \|\mathbf{u} - \mathbf{u}^r_{mn}\|_H^2(t)\,\mathrm{d}t.$$

Proof. We make repeated use of the inequality

$$ab \leqslant \frac{1}{4d}a^2 + db^2 \tag{19}$$

that holds for any positive d.

(a) $\|\mathbf{u}^0 - \mathbf{P}^r_{mn}\mathbf{u}^0\|_H \geqslant \|\mathbf{u}^0 - \mathbf{P}^r_{mn}\mathbf{u}^0\|_0$. Our choice of initial condition $\mathbf{u}^{r0}_{mn}$ in terms of the L_2 orthonormal basis $\{\mathscr{B}_i\}$ minimizes $\|\mathbf{u}^0 - \mathbf{v}\|_0$ over $\mathbf{v} \in \mathbf{G}_0([n], m)$. Therefore $\|\mathbf{u}^0 - \mathbf{P}^r_{mn}\mathbf{u}^0\|_0 \geqslant \|\mathbf{u}^0 - \mathbf{u}^{r0}_{mn}\|_0$ and the result follows from Lemma 1.1.

(b) Both $\mathbf{u}$ and $\mathbf{u}^r_{mn}$ are sufficiently smooth so we can integrate the left-hand side of (b) by parts with respect to t:

$$\int_0^\tau (\mathbf{u}' - \mathbf{u}^r_{mn}{}', \mathbf{u} - \mathbf{P}^r_{mn}\mathbf{u})_0\,\mathrm{d}t$$

$$= \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u} - \mathbf{u}^r_{mn}, \mathbf{u} - \mathbf{P}^r_{mn}\mathbf{u})_0\,\mathrm{d}t - \int_0^\tau \left(\mathbf{u} - \mathbf{u}^r_{mn}, \frac{\partial}{\partial t}(\mathbf{u} - \mathbf{P}^r_{mn}\mathbf{u})\right)_0 \mathrm{d}t$$

$$= (\mathbf{u}(\tau) - \mathbf{u}^r_{mn}(\tau), \mathbf{u}(\tau) - \mathbf{P}^r_{mn}\mathbf{u}(\tau))_0 - (\mathbf{u}^0 - \mathbf{u}^{r0}_{mn}, \mathbf{u}^0 - \mathbf{P}^r_{mn}\mathbf{u}^0)_0 - \int_0^\tau \left(\mathbf{u} - \mathbf{u}^r_{mn}, \frac{\partial}{\partial t}(\mathbf{u} - \mathbf{P}^r_{mn}\mathbf{u})\right)_0 \mathrm{d}t.$$

Then

$$\left|\int_0^\tau (\mathbf{u}' - \mathbf{u}^r_{mn}{}', \mathbf{u} - \mathbf{P}^r_{mn}\mathbf{u})\,\mathrm{d}t\right|$$

$$\leqslant \|\mathbf{u}(\tau) - \mathbf{u}^r_{mn}(\tau)\|_0 \|\mathbf{u}(\tau) - \mathbf{P}^r_{mn}\mathbf{u}(\tau)\|_0 + \|\mathbf{u}^0 - \mathbf{u}^{r0}_{mn}\|_0 \|\mathbf{u}^0 - \mathbf{P}^r_{mn}\mathbf{u}^0\|_0 + \int_0^\tau \|\mathbf{u} - \mathbf{u}^r_{mn}\|_0 \left\|\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{P}^r_{mn}\mathbf{u})\right\|_0 \mathrm{d}t.$$

Now $\partial/\partial t(\mathbf{u} - \mathbf{P}^r_{mn}\mathbf{u}) = \partial\mathbf{u}/\partial t - \mathbf{P}^r_{mn}(\partial\mathbf{u}/\partial t)$ since we can show that $\partial/\partial t(\mathbf{P}^r_{mn}\mathbf{u}) = \mathbf{P}^r_{mn}(\partial\mathbf{u}/\partial t)$ by considering the representation of $\mathbf{u}$ in $\mathbf{G}_0[\mathbf{m}][\mathbf{n}][\mathbf{r}]$ as a finite sum of $\mathbf{H}$-orthonormal basis functions. The result then follows from (19).

(c) We use (19) and the inequality

$$|\mathbf{a}(\mathbf{u}-\mathbf{u}^r_{mn},\mathbf{u}-\mathbf{P}^r_{mn}\mathbf{u})|(\tau)\leqslant\|\mathbf{u}(\tau)-\mathbf{u}^r_{mn}(\tau)\|_H\cdot\|\mathbf{u}(\tau)-\mathbf{P}^r_{mn}\mathbf{u}(\tau)\|_H$$

to obtain a proof of (c).

(d) On each cell Ω_i, with v_{ij} denoting the outward normal to Γ_{ij},

$$-(\operatorname{grad}\Pi,\delta)_{0,i}=(\Pi,\nabla\cdot\delta)_{0,i}-\sum_j\int_{\Gamma_{ij}}\sum_{q=1}^{K}\gamma_{ij}(\Pi\delta^q)v^q_{ij}\,\mathrm{d}s.$$

We add these representations for all cells. Since $\gamma_{ij}(\Pi)=\gamma_{ji}(\Pi)$ and the outward unit normal vector $v^{\cdot}_{ij}$ on Γ_{ij} for Ω_i equals $-v^{\cdot}_{ji}$, the outward normal to Γ_{ij} for Ω_j, we can combine terms to obtain

$$\begin{aligned}-(\operatorname{grad}\Pi,\delta)_0=(\Pi,\nabla\cdot\delta)_0&-\sum_{\Gamma_{ij}}\int_{\Gamma_{ij}}\sum_{q=1}^{K}\gamma_{ij}(\Pi)[\gamma_{ij}(\delta^q)-\gamma_{ji}(\delta^q)]v^q_{ij}\,\mathrm{d}s\\&-\sum_{\Gamma_{i0}}\int_{\Gamma_{i0}}\sum_{q=1}^{K}\gamma_{i0}(\Pi)[\gamma_{i0}(\delta^q)-0]v^q_{i0}\,\mathrm{d}s.\end{aligned}\tag{20}$$

On each cell $\Omega_k,\Pi=S^k_r(\Pi)+\mathscr{T}_r(\Pi)$, where $S^k_r(\Pi)$ is the H-orthogonal projection of Π onto the span of $\{B^k_i\}^r_{i=1}$. Since $\delta\in\mathbf{G}^T_0[\mathbf{m}][\mathbf{n}][\mathbf{r}]$, $\nabla\cdot\delta$ is orthogonal to $S^k_r(\Pi)$, so

$$|(\Pi,\nabla\cdot\delta)_{0,k}|=|(\mathscr{T}_r(\Pi),\nabla\cdot\delta)_{0,k}|.$$

Then Schwarz' inequality gives $|(\Pi,\nabla\cdot\delta)_0|\leqslant\|\mathscr{T}_r(\Pi)\|_0\|\nabla\cdot\delta\|_0\leqslant\sqrt{K}\|\mathscr{T}_r(\Pi)\|_0\|\delta\|_H$.

We obtain estimates for the two sums in (20) in a similar manner. For each Γ_{ij} and each q, $\gamma_{ij}(\Pi)v^q_{ij}=S^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^q_{ij})+\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^q_{ij})$, where $S^{ij}_{n_{ij}}(v)$ is the $L_2(\Gamma_{ij})$ projection of v onto the span of $\{\omega^{ij}_k\}^{n_{ij}}_{k=1}$. Since $\delta\in\mathbf{G}^T_0[\mathbf{m}][\mathbf{n}][\mathbf{r}]$, the moment collocation requirements give

$$-\int_{\Gamma_{ij}}\sum_{q=1}^{K}\gamma_{ij}(\Pi)[\gamma_{ij}(\delta^q)-\gamma_{ji}(\delta^q)]v^q_{ij}\,\mathrm{d}s=-\int_{\Gamma_{ij}}\sum_{q=1}^{K}\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^q_{ij})[\gamma_{ij}(\delta^q)-\gamma_{ji}(\delta^q)]\,\mathrm{d}s.$$

If we use $\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^{\cdot}_{ij})$ to represent the vector function defined on Γ_{ij} with qth component $\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^q_{ij})$ then

$$\left|\int_{\Gamma_{ij}}\sum_{q=1}^{K}\gamma_{ij}(\Pi)[\gamma_{ij}(\delta^q)-\gamma_{ji}(\delta^q)]v^q_{ij}\,\mathrm{d}s\right|\leqslant\|\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^{\cdot}_{ij})\|_{ij}\|\gamma_{ij}(\delta)-\gamma_{ji}(\delta)\|_{ij}.\tag{21}$$

A similar argument gives us the estimate

$$\left|\int_{\Gamma_{i0}}\sum_{q=1}^{K}\gamma_{i0}(\Pi)[\gamma_{i0}(\delta^q)-0]v^q_{i0}\,\mathrm{d}s\right|\leqslant\|\mathscr{T}^{i0}_{n_{i0}}(\gamma_{i0}(\Pi)v^{\cdot}_{i0})\|_{i0}\|\gamma_{i0}(\delta)\|_{i0}.\tag{22}$$

By the trace theorem, using the constants C_{ij},

$$\|\gamma_{ji}(\delta)-\gamma_{ij}(\delta)\|_{ij} \leqslant \|\gamma_{ji}(\delta)\|_{ij}+\|\gamma_{ij}(\delta)\|_{ij} \leqslant C_{ji}\|\delta\|_{1,j}+C_{ij}\|\delta\|_{1,i}$$

and

$$\|\gamma_{i0}(\delta)\|_{i0} \leqslant C_{i0}\|\delta\|_{1,i}.$$

By the argument in [23], since the $\|\delta\|_{1,i}$ occur at most n_f times in the sums of the expressions in (21) and (22) over the Γ_{ij},

$$\left|\sum_{\Gamma_{ij}}\int_{\Gamma_{ij}}\sum_{q=1}^{K}\gamma_{ij}(\Pi)[\gamma_{ij}(\delta^q)-\gamma_{ji}(\delta^q)]v_{ij}^{q}\,\mathrm{d}s+\sum_{\Gamma_{i0}}\int_{\Gamma_{i0}}\sum_{q=1}^{K}\gamma_{i0}(\Pi)[\gamma_{i0}(\delta^q)-0]v_{i0}^{q}\,\mathrm{d}s\right|$$

$$\leqslant \sup\{C_{ij}\}\sup\{\|\mathscr{T}_{n_{ij}}^{ij}(\gamma_{ij}(\Pi)v_{ij}^{\cdot})\|_{ij}\}n_f\sqrt{N}\|\delta\|_H.$$

We collect these estimates and integrate over time to obtain

$$\left|\int_0^\tau(\nabla\Pi,\delta)_0\,\mathrm{d}t\right| \leqslant \int_0^\tau[\sqrt{K}\|\mathscr{T}_r(\Pi)\|_0+\sup\{C_{ij}\}\sup\{\|\mathscr{T}_{n_{ij}}^{ij}(\gamma_{ij}(\Pi)v_{ij}^{\cdot})\|_{ij}\}n_f]\|\delta\|_H\,\mathrm{d}t.$$

The result follows from (19) (used twice).

(e) From [23], since $\delta\in\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}][\mathbf{r}],\gamma_{ij}(\delta)-\gamma_{ji}(\delta)$ is orthogonal to the $[n]$ collocation functions, we use Schwarz' inequality and the trace theorem to obtain estimate

$$\left|\sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\delta)-\gamma_{ji}(\delta)\rangle_{ij}+\sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_{i0}}\mathbf{u},\gamma(\delta)\rangle_{i0}\right|$$

$$=\left|\sum_{\Gamma_{i}}\langle\mathscr{T}_{n_{ij}}^{ij}(D_{\mathbf{n}_{ij}}\mathbf{u}),\gamma_{ij}(\delta)-\gamma_{ji}(\delta)\rangle_{ij}+\sum_{\Gamma_{i0}}\langle\mathscr{T}_{n_{i0}}^{i0}(D_{\mathbf{n}_{i0}}\mathbf{u}),\gamma(\delta)\rangle_{i0}\right|$$

$$\leqslant \sup\{C_{ij}\}\sup\{\|\mathscr{T}_{n_{ij}}^{ij}(D_{\mathbf{n}_{ij}}\mathbf{u})\|_{ij}\}(n_f)\sqrt{N}\|\delta\|_H.$$

When we integrate this expression over $[0,\tau]$, we obtain the estimate

$$\left|\int_0^\tau\left[\sum_{\Gamma_{ij}}\langle D_{\mathbf{n}_{ij}}\mathbf{u},\gamma_{ij}(\mathbf{u}_{mn}^r)-\gamma_{ji}(\mathbf{u}_{mn}^r)\rangle_{ij}+\sum_{\Gamma_{i0}}\langle D_{\mathbf{n}_i}\mathbf{u},\gamma(\mathbf{u}_{mn}^r)\rangle_{i0}\right]\,\mathrm{d}t\right|$$

$$\leqslant \int_0^\tau\sqrt{N}n_f\sup\{C_{ij}\}\sup\{\|\mathscr{T}_{n_{ij}}^{ij}(D_{\mathbf{n}_{ij}}\mathbf{u})\|_{ij}\}\|\mathbf{u}-\mathbf{u}_{mn}^r\|_H(t)\,\mathrm{d}t.$$

The result follows from inequality (19). □

Proof of Theorem 1.2. Using Lemma 1.3, we obtain the following estimate for Eq. (18):

$$\frac{1}{2}\|\mathbf{u}(\tau)-\mathbf{u}_{mn}^r(\tau)\|_0^2+c\int_0^\tau\|\mathbf{u}-\mathbf{u}_{mn}^r\|_H^2(t)\,\mathrm{d}t$$

$$\leqslant \frac{1}{2}\|\mathbf{u}^0-\mathbf{u}_{mn}^{r0}\|_0^2+\frac{1}{4}\|\mathbf{u}(\tau)-\mathbf{u}_{mn}^r(\tau)\|_0^2+\|\mathbf{u}(\tau)-\mathbf{P}_{mn}^r\mathbf{u}(\tau)\|_0^2+\frac{1}{2}\|\mathbf{u}^0-\mathbf{u}_{mn}^{r0}\|_0^2$$

$$+\frac{1}{2}\|\mathbf{u}^0-\mathbf{P}^r_{mn}\mathbf{u}^0\|_0^2+\frac{3c}{20}\int_0^\tau\|\mathbf{u}-\mathbf{u}^r_{mn}\|_H^2\,\mathrm{d}t+\frac{5}{3c}\int_0^\tau\|\mathbf{u}'-\mathbf{P}^r_{mn}\mathbf{u}'\|_0^2\,\mathrm{d}t$$

$$+\frac{3c}{20}\int_0^\tau\|\mathbf{u}-\mathbf{u}^r_{mn}\|_H^2\,\mathrm{d}t+\frac{5}{3c}\int_0^\tau\|\mathbf{u}-\mathbf{P}^r_{mn}\mathbf{u}\|_H^2\,\mathrm{d}t+\int_0^\tau\frac{5}{3c}[K\|\mathscr{T}_r(\Pi)\|_0^2$$

$$+Nn_f^2\sup\{C_{ij}^2\}\sup\{\|\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^{\cdot}_{ij})\|_{ij}^2\}]+\frac{6c}{20}\|\mathbf{u}-\mathbf{u}^r_{mn}\|_H^2\,\mathrm{d}t$$

$$+\frac{5}{3c}Nn_f^2\sup\{C_{ij}^2\}\int_0^\tau\sup\{\|\mathscr{T}^{ij}_{n_{ij}}(D_{\mathbf{n}_i}\mathbf{u})\|_{ij}^2\}\,\mathrm{d}t+\frac{3c}{20}\int_0^\tau\|\mathbf{u}-\mathbf{u}^r_{mn}\|_H^2\,\mathrm{d}t.$$

Consolidating similar terms, we get

$$\frac{1}{4}\|\mathbf{u}(\tau)-\mathbf{u}^r_{mn}(\tau)\|_0^2+\frac{c}{4}\int_0^\tau\|\mathbf{u}-\mathbf{u}^r_{mn}\|_H^2(t)\,\mathrm{d}t$$

$$\leqslant\frac{1}{2}\|\mathbf{u}^0-\mathbf{u}^{r0}_{mn}\|_0^2+\|\mathbf{u}(\tau)-\mathbf{P}^r_{mn}\mathbf{u}(\tau)\|_0^2+\frac{1}{2}\|\mathbf{u}^0-\mathbf{u}^{r0}_{mn}\|_0^2+\frac{1}{2}\|\mathbf{u}^0-\mathbf{P}^r_{mn}\mathbf{u}^0\|_0^2$$

$$+\frac{5}{3c}\int_0^\tau[\|\mathbf{u}'-\mathbf{P}^r_{mn}\mathbf{u}'\|_0^2+\|\mathbf{u}-\mathbf{P}^r_{mn}\mathbf{u}\|_H^2+K\|\mathscr{T}_r(\Pi)\|_0^2$$

$$+Nn_f^2\sup\{C_{ij}^2\}(\sup\{\|\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^{\cdot}_{ij})\|_{ij}^2\}+\sup\{\|\mathscr{T}^{ij}_{n_{ij}}(D_{\mathbf{n}_i})\|_{ij}^2\})]\,\mathrm{d}t.$$

The theorem then follows from Lemmas 1.1 and 1.3(a). □

We use Dini's theorem to argue that this theorem gives a general convergence result.

With our assumptions concerning the smoothness of $\mathbf{u}$ and $\mathbf{u}'$, since $\|\mathscr{T}^{ij}_{n_{ij}}(D_{\mathbf{n}_i}\mathbf{u})\|_{ij}^2(t)$ and $\|\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^{\cdot}_{ij})\|_{ij}^2(t)$ are continuous in t and go monotonically to 0 as $[n]\to[\infty]$ for fixed t by the properties of $\mathscr{T}^{ij}_{n_{ij}}$, these terms can be made sufficiently small uniformly in t for sufficiently large $[n]$. Fix such an $[n]$. Likewise $\|\mathscr{T}_r(\Pi)\|_0^2(t)$ can be made small uniformly in t for sufficiently large $[r]$. With $[n]$ and $[r]$ so chosen, Lemma 1.1 gives a bound for $K(n,r)$.

Thus, with $[n]$ and $[r]$ fixed so that the terms involving Π and the normal derivative of $\mathbf{u}$ are sufficiently small, choose $[m]$ large enough so that the entire error is small uniformly in t by using Dini's theorem once again. This argument establishes convergence in $C([0,T];\mathbf{L}_2(\Omega))$ as well as $L_2([0,T];\mathbf{H}(\Omega))$.

We obtain an approximation to the pressure in the following way. The procedures are discussed in more detail in [25].

Eq. (15) established that $\mathbf{b}'+\mathbf{C}\mathbf{b}-((\mathbf{f},\Phi^j)_0)$ is orthogonal to the null space of $\mathscr{M}$. Then there exists some vector $\mathbf{h}$ such that $\mathbf{b}'+\mathbf{C}\mathbf{b}-((\mathbf{f},\Phi^j))=\mathscr{M}^{\mathrm{T}}\mathbf{h}$. We obtain $\mathbf{h}$ as follows. The QR representation of $\mathscr{M}^{\mathrm{T}}$ is $(Q''|Q')\binom{\mathbf{R}}{\mathbf{O}}$; we have argued that the columns of $\mathbf{Q}'$ span the null space of $\mathscr{M}$. Hence the (orthonormal) columns of $\mathbf{Q}''$ span the orthogonal complement of the null space of $\mathscr{M}$, which contains $\mathbf{b}'+\mathbf{C}\mathbf{b}-((\mathbf{f},\Phi^j)_0)$. One readily verifies that $\mathbf{h}$ can be

$$\mathbf{R}^{-1}(\mathbf{Q}'')^{\mathrm{T}}(\mathbf{b}'+\mathbf{C}\mathbf{b}-((\mathbf{f},\Phi^j)))=\mathbf{R}^{-1}(\mathbf{Q}'')^{\mathrm{T}}(\mathbf{C}\mathbf{b}-((\mathbf{f},\Phi^j))),$$

since $\mathbf{b}'(t) = \mathbf{Q}'\mathbf{y}'(t)$ and $(\mathbf{Q}'')^{\mathrm{T}}\mathbf{Q}' = \mathbf{0}$. Thus we obtain relation

$$\mathbf{b}' + \mathbf{C}\mathbf{b} - \mathscr{M}^{\mathrm{T}}\mathbf{h} = ((\mathbf{f}, \Phi^j)_0). \tag{23}$$

In the two-dimensional case, the matrix $\mathscr{M}^{\mathrm{T}}$ has the following form:

$$\mathscr{M}^{\mathrm{T}} = \begin{pmatrix} \mathbf{M}_1^{\mathrm{T}} & \mathbf{0} & \mathbf{S}_1^{\mathrm{T}} \\ \mathbf{0} & \mathbf{M}_2^{\mathrm{T}} & \mathbf{S}_2^{\mathrm{T}} \end{pmatrix}. \tag{24}$$

We represent $\mathbf{h}$ by $(\lambda_1(t), \lambda_2(t), \pi(t))^{\mathrm{T}}$, where vector λ_i^{T} is that part of $\mathbf{h}$ that is multiplied by $\mathbf{M}_i^T$, $i = 1, 2$, and π consists of the last Nr components of $\mathbf{h}$. Matrix $\mathbf{C}$ consists of two identical blocks denoted $\mathbf{C}_i, i = 1, 2$. We represent the two components of $((\mathbf{f}, \Phi^j)_0)$ as $(\mathbf{f}_1, \mathbf{f}_2)^{\mathrm{T}}$.

Then the components $(\mathbf{b}_1, \mathbf{b}_2)^{\mathrm{T}}(t)$ of $\mathbf{b}^{\mathrm{T}}$ satisfy

$$\mathbf{b}_1' + \mathbf{C}_1\mathbf{b}_1 - \mathbf{M}_1^{\mathrm{T}}\lambda_1^{\mathrm{T}} - \mathbf{S}_1^{\mathrm{T}}\pi^{\mathrm{T}} = \mathbf{f}_1$$

and

$$\mathbf{b}_2' + \mathbf{C}_2\mathbf{b}_2 - \mathbf{M}_2^{\mathrm{T}}\lambda_2^{\mathrm{T}} - \mathbf{S}_2^{\mathrm{T}}\pi^{\mathrm{T}} = \mathbf{f}_2.$$

Members of the space $\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}] \equiv C([0, T]; \mathbf{G}_0[\mathbf{m}][\mathbf{n}])$ (in our two-dimensional representation) correspond to vector pairs $(\mathbf{a}_1, \mathbf{a}_2)$ whose components satisfy $\mathbf{M}_i\mathbf{a}_i^{\mathrm{T}} = 0$. If we multiply the vector equations above by row vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, we get

$$\mathbf{a}_1\mathbf{b}_1' + \mathbf{a}_1\mathbf{C}_1\mathbf{b}_1 - \mathbf{a}_1\mathbf{S}_1^{\mathrm{T}}\pi^{\mathrm{T}} = \mathbf{a}_1\mathbf{f}_1$$

and

$$\mathbf{a}_2\mathbf{b}_2' + \mathbf{a}_2\mathbf{C}_2\mathbf{b}_2 - \mathbf{a}_2\mathbf{S}_2^{\mathrm{T}}\pi^{\mathrm{T}} = \mathbf{a}_2\mathbf{f}_2. \tag{25}$$

Vector $\pi(t)$ has size Nr; we represent it as $(\pi_1^1, \pi_2^1, \ldots, \pi_r^1, \pi_1^2, ., .., \pi_r^N)$. Our approximate pressure, denoted Π_{mn}^r, is defined on each of the N cells Ω_k to be

$$\Pi_{mn}^r|_{\Omega_k} = \sum_{i=1}^{r} \pi_i^k(t) B_i^k. \tag{26}$$

Π_{mn}^r is in space $C([0, t]; H)$.

Then Eq. (25) are equivalent to the result that for all $\mathbf{w} \in \mathbf{G}_0^T[\mathbf{m}][\mathbf{n}]$,

$$(\mathbf{u}_{mn}^{r'}, \mathbf{w})_0 + \mathbf{a}(\mathbf{u}_{mn}^r, \mathbf{w}) - (\nabla \cdot \mathbf{w}, \Pi_{mn}^r)_0 = (\mathbf{f}, \mathbf{w})_0. \tag{27}$$

We obtain the following results concerning convergence of this approximation to the pressure.

Theorem 1.4. *We use the notation and assumptions of Theorem* 1.2. *For any* $\mathbf{w} \in \mathbf{G}_0^T[\mathbf{m}][\mathbf{n}]$,

$$\left| \int_0^T (\Pi_{mn}^r - \Pi, \nabla \cdot \mathbf{w})_0 \, \mathrm{d}t \right|$$
$$\leqslant \|\mathbf{u}_{mn}^r(T) - \mathbf{u}(T)\|_0 \|\mathbf{w}(T)\|_0 + \|\mathbf{u}_{mn}^{r0} - \mathbf{u}^0\|_0 \|\mathbf{w}(0)\|_0$$

$$+\left(\int_0^T \|\mathbf{u}-\mathbf{u}_{mn}^r\|_0^2\,\mathrm{d}t\right)^{1/2}\left(\int_0^T \|\mathbf{w}'\|_0^2\,\mathrm{d}t\right)^{1/2}$$

$$+\left[\left(\int_0^T \|\mathbf{u}-\mathbf{u}_{mn}^r\|_H^2\,\mathrm{d}t\right)^{1/2}\sup\{C_{ij}\}(n_f)\sqrt{N}\left\{\left(\int_0^T \sup\{\|\mathscr{T}_{n_{ij}}^{ij}(D_{\mathbf{n}_{ij}}\mathbf{u})\|_{ij}^2\}\,\mathrm{d}t\right)^{1/2}\right.\right.$$

$$\left.\left.+\left(\int_0^T \sup\{\|\mathscr{T}_{n_{ij}}^{ij}(\gamma_{ij}(\Pi)v_{ij}^{\cdot})\|_{ij}^2\}\,\mathrm{d}t\right)^{1/2}\right\}\right]\left(\int_0^T \|\mathbf{w}\|_H^2\,\mathrm{d}t\right)^{1/2}.$$

Proof. From (27), for any $\mathbf{w}\in\mathbf{G}_0^T[\mathbf{m}][\mathbf{n}]$,

$$(\nabla\cdot\mathbf{w},\Pi_{mn}^r)_0=(\mathbf{u}_{mn}^{r'},\mathbf{w})_0+\mathbf{a}(\mathbf{u}_{mn}^r,\mathbf{w})-(\mathbf{f},\mathbf{w})_0.$$

From (1) and (17),

$$(\operatorname{grad}\Pi,\mathbf{w})_0=-(\mathbf{u}',\mathbf{w})_0-\mathbf{a}(\mathbf{u},\mathbf{w})+(\mathbf{f},\mathbf{w})_0$$
$$+\sum_{\Gamma_{ij}}\langle\mathscr{T}_{n_{ij}}^{ij}(D_{v_{ij}}\mathbf{u}),\gamma_{ij}(\mathbf{w})-\gamma_{ji}(\mathbf{w})\rangle_{ij}+\sum_{\Gamma_{i0}}\langle\mathscr{T}_{n_{i0}}^{i0}(D_{v_{ij}}\mathbf{u}),\gamma_{i0}(\mathbf{w})\rangle_{i0}.$$

On the other hand, from (20) in Lemma 1.3,

$$(\operatorname{grad}\Pi,\mathbf{w})_0=-(\Pi,\nabla\cdot\mathbf{w})_0+\sum_{\Gamma_{ij}}\int_{\Gamma_{ij}}\sum_{q=1}^K\gamma_{ij}(\Pi)[\gamma_{ij}(\mathbf{w}^q)-\gamma_{ji}(\mathbf{w}^q)]v_{ij}^q\,\mathrm{d}s$$
$$+\sum_{\Gamma_{i0}}\int_{\Gamma_{i0}}\sum_{q=1}^K\gamma_{i0}(\Pi)[\gamma_{i0}(\mathbf{w}^q)-0]v_{i0}^q\,\mathrm{d}s.$$

Then $(\nabla\cdot\mathbf{w},\Pi_{mn}^r-\Pi)_0=(\mathbf{u}_{mn}^{r'},\mathbf{w})_0+\mathbf{a}(\mathbf{u}_{mn}^r,\mathbf{w})-(\mathbf{f},\mathbf{w})_0+(\operatorname{grad}\Pi,\mathbf{w})_0$

$$-\sum_{\Gamma_{ij}}\int_{\Gamma_{ij}}\sum_{q=1}^K\gamma_{ij}(\Pi)[\gamma_{ij}(\mathbf{w}^q)-\gamma_{ji}(\mathbf{w}^q)]v_{ij}^q\,\mathrm{d}s-\sum_{\Gamma_{i0}}\int_{\Gamma_{i0}}\sum_{q=1}^K\gamma_{i0}(\Pi)[\gamma_{i0}(\mathbf{w}^q)-0]v_{i0}^q\,\mathrm{d}s$$

$$=(\mathbf{u}_{mn}^{r'},\mathbf{w})_0+\mathbf{a}(\mathbf{u}_{mn}^r,\mathbf{w})-(\mathbf{f},\mathbf{w})_0-(\mathbf{u}',\mathbf{w})_0-\mathbf{a}(\mathbf{u},\mathbf{w})+(\mathbf{f},\mathbf{w})_0$$
$$+\sum_{\Gamma_{ij}}\langle\mathscr{T}_{n_{ij}}^{ij}(D_{v_{ij}}\mathbf{u}),\gamma_{ij}(\mathbf{w})-\gamma_{ji}(\mathbf{w})\rangle_{ij}+\sum_{\Gamma_{i0}}\langle\mathscr{T}_{n_{i0}}^{i0}(D_{v_{i0}}\mathbf{u}),\gamma_{i0}(\mathbf{w})\rangle_{i0}$$
$$-\sum_{\Gamma_{ij}}\int_{\Gamma_{ij}}\sum_{q=1}^K\gamma_{ij}(\Pi)[\gamma_{ij}(\mathbf{w}^q)-\gamma_{ji}(\mathbf{w}^q)]v_{ij}^q\,\mathrm{d}s-\sum_{\Gamma_{i0}}\int_{\Gamma_{i0}}\sum_{q=1}^K\gamma_{i0}(\Pi)[\gamma_{i0}(\mathbf{w}^q)-0]v_{i0}^q\,\mathrm{d}s$$

$$
\begin{aligned}
&= (\mathbf{u}_{mn}^{r'} - \mathbf{u}', \mathbf{w})_0 + \mathbf{a}(\mathbf{u}_{mn}^r - \mathbf{u}, \mathbf{w}) \\
&\quad + \sum_{\Gamma_{ij}} \langle \mathscr{T}_{n_{ij}}^{ij}(D_{v_{ij}}\mathbf{u}), \gamma_{ij}(\mathbf{w}) - \gamma_{ji}(\mathbf{w})\rangle_{ij} + \sum_{\Gamma_{i0}} \langle \mathscr{T}_{n_{i0}}^{i0}(D_{v_{i0}}\mathbf{u}), \gamma_{i0}(\mathbf{w})\rangle_{i0} \\
&\quad - \sum_{\Gamma_{ij}} \int_{\Gamma_{ij}} \sum_{q=1}^{K} \gamma_{ij}(\Pi)[\gamma_{ij}(\mathbf{w}^q) - \gamma_{ji}(\mathbf{w}^q)] v_{ij}^q \, \mathrm{d}s - \sum_{\Gamma_{i0}} \int_{\Gamma_{i0}} \sum_{q=1}^{K} \gamma_{i0}(\Pi)[\gamma_{i0}(\mathbf{w}^q) - 0] v_{i0}^q \, \mathrm{d}s.
\end{aligned}
$$

We integrate this expression from 0 to T and use the estimates derived in the proof of Lemma 1.3 to majorize the sums over the Γ_{ij} to obtain

$$
\begin{aligned}
\left| \int_0^T (\Pi_{mn}^r - \Pi, \nabla \cdot \mathbf{w})_0 \, \mathrm{d}t \right| &\leqslant \left| \int_0^T (\mathbf{u}_{mn}^{r'} - \mathbf{u}', \mathbf{w})_0 \, \mathrm{d}t \right| + \left| \int_0^T \|\mathbf{u}_{mn}^r - \mathbf{u}\|_H \|\mathbf{w}\|_H \, \mathrm{d}t \right| \\
&\quad + \sup\{C_{ij}\}(n_f)\sqrt{N} \int_0^T (\sup\{\|\mathscr{T}_{n_{ij}}^{ij}(D_{\mathbf{n}_{ij}}\mathbf{u})\|_{ij}\} \\
&\quad + \sup\{\|\mathscr{T}_{n_{ij}}^{ij}(\gamma_{ij}(\Pi)v_{ij}^{\cdot})\|_{ij}\})\|\mathbf{w}\|_H \, \mathrm{d}t.
\end{aligned} \tag{28}
$$

We integrate the first term on the right side of inequality (28) by parts in variable t as in Lemma 1.3(b) to get

$$
\begin{aligned}
&\left| \int_0^T (\mathbf{u}_{mn}^{r}{}' - \mathbf{u}', \mathbf{w})_0 \, \mathrm{d}t \right| \\
&\quad \leqslant \|\mathbf{u}_{mn}^r(T) - \mathbf{u}(T)\|_0 \|\mathbf{w}(T)\|_0 + \|\mathbf{u}_{mn}^{r0} - \mathbf{u}^0\|_0 \|\mathbf{w}(0)\|_0 + \int_0^T \|\mathbf{u} - \mathbf{u}_{mn}^r\|_0 \|\mathbf{w}'\|_0 \, \mathrm{d}t.
\end{aligned}
$$

The result follows from Schwarz' inequality in $L_2[0,T]$. □

We can assess the effectiveness of the estimate in Theorem 1.4 if we consider the following semi-norms.

Definition 1.5. For any $v \in C([0,T]; L_2(\Omega))$, let

$$
\begin{aligned}
|v|_{0[m][n]}^{\mathrm{T}} \equiv \sup \Bigg\{ &\int_0^T (v, \nabla \cdot \mathbf{w})_0 \, \mathrm{d}t \colon \mathbf{w} \in C^1([0,T]; \mathbf{G}_0[\mathbf{m}][\mathbf{n}]) \text{ and} \\
&\max\left\{ \|\mathbf{w}(T)\|_0, \|\mathbf{w}(0)\|_0, \int_0^T \|\mathbf{w}'\|_0^2 \, \mathrm{d}t, \int_0^T \|\mathbf{w}\|_H^2 \, \mathrm{d}t \right\} \leqslant 1 \Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } |\mathbf{v}|_0^{\mathrm{T}} \equiv \sup \Bigg\{ &\int_0^T (v, \nabla \cdot \mathbf{w})_0 \, \mathrm{d}t \colon \mathbf{w} \in C([0,T]; \mathbf{H}_0^1(\mathbf{\Omega})) \cap C^1([0,T]; \mathbf{L}_2(\Omega)) \text{ and} \\
&\max\left\{ \|\mathbf{w}(T)\|_0, \|\mathbf{w}(0)\|_0, \int_0^T \|\mathbf{w}'\|_0^2 \, \mathrm{d}t, \int_0^T \|\mathbf{w}\|_H^2 \, \mathrm{d}t \right\} \leqslant 1 \Bigg\}.
\end{aligned}
$$

We assemble the results of Theorems 1.2 and 1.4. We use notation $\|\cdot\|_H^T$ to denote the $L_2([0,T]\colon \mathbf{H})$ norm $(\int_0^T \|\cdot\|_H^2\,\mathrm{d}t)^{1/2}$.

Theorem 1.6. *We use the notations and assumptions of Theorem* 1.2.

$$
\begin{aligned}
&|\Pi^r_{mn} - \Pi|^T_{0[m][n]}\\
&\quad\leqslant (1+2c^{-1/2})\left[3c^{-1/2}\left(\int_0^T \|\mathscr{T}_r(\Pi)\|_0^2\,\mathrm{d}t\right)^{1/2}\right.\\
&\qquad\left. +K(n,r)(3\|\mathbf{Q}_m\mathbf{u}^0\|_H + 2\|\mathbf{Q}_m\mathbf{u}(T)\|_H + 3c^{-1/2}(\|\mathbf{Q}_m\mathbf{u}\|_H^T + \|\mathbf{Q}_m\mathbf{u}'\|_H^T))\right]\\
&\qquad + (1+3c^{-1/2}(1+2c^{-1/2}))\sup\{C_{ij}\}(n_f)\sqrt{N}\\
&\qquad\times\left[\left(\int_0^T \sup\{\|\mathscr{T}^{ij}_{n_{ij}}(D_{\mathbf{n}_{ij}}\mathbf{u})\|_{ij}^2\}\,\mathrm{d}t\right)^{1/2} + \left(\int_0^T \sup\{\|\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v_{ij}^{\cdot})\|_{ij}^2\}\,\mathrm{d}t\right)^{1/2}\right].
\end{aligned}
$$

As in the discussion following Theorem 1.2, we can use Dini's theorem to argue that we can make $|\Pi^r_{mn} - \Pi|^T_{0[m][n]}$ arbitrarily small by first taking $[n]$ and $[r]$ sufficiently large to make the terms of the estimate that depend only on $[n]$ and $[r]$ sufficiently small. Then we obtain a bound for $K(n,r)$ and we can make the rest of the terms small by making $[m]$ suitably large. *Note that the estimate remains small as we then increase* $[m]$. However, $|\cdot|^T_{0[m][n]}$ is an awkward measure. We can clarify its meaning somewhat with the following lemma.

Lemma 1.7. *For any* $v\in C([0,T];L_2(\Omega))$,

(a) $[m']\leqslant[m]$ *and* $[n']\geqslant[n] \Rightarrow |v|^T_{0[m'][n']} \leqslant |v|^T_{0[m][n]}$;

(b) $|v|^T_{0[m][n]} \leqslant \sqrt{K}(\int_0^T \|v\|_0^2\,\mathrm{d}t)^{1/2}$.

(c) *If* $[n]$ *has the property that for any* Γ_{ij}, *the constant function is in the span of* $\{\omega^{ij}_k\colon 1\leqslant k\leqslant n_{ij}\}$ *and the cells are polyhedral*, *then if* $v(x,t)$ *is independent of* x, $|v|^T_{0[m][n]} = 0$.

(d) $|\cdot|_0^T$ *is a norm for* $C([0,T];H^1_0(\Omega))$.

(e) *For any* $v\in C([0,T];H^1(\Omega)), |v|_0^T = 0$ *if and only if*, *for each* $t\in[0,T], v(t)$ *is constant a.e. with respect to* x.

Proof. Part (a) follows from the definition of $|\cdot|^T_{0[m][n]}$ and the inclusions $[m']\leqslant[m]$ and $[n']\geqslant[n]\Rightarrow \mathbf{G}_0[\mathbf{m}'][\mathbf{n}'] \subset \mathbf{G}_0[\mathbf{m}][\mathbf{n}]$. Schwarz' inequality and the inequality $\|\nabla\cdot\mathbf{w}\|_0 \leqslant \sqrt{K}\|\mathbf{w}\|_H$ yield (b).

To show (c), if v is independent of x, we use the divergence theorem on each cell to get

$$
\begin{aligned}
\int\int_\Omega v(t)\nabla\cdot\mathbf{w}\,\mathrm{d}x &= v(t)\sum_{i=1}^N \int\int_{\Omega_i} \nabla\cdot\mathbf{w}\,\mathrm{d}x\\
&= v(t)\sum_{\Gamma_{ij}}\int_{\Gamma_{ij}} (\gamma_{ij}(\mathbf{w}) - \gamma_{ji}(\mathbf{w}))\cdot v_{ij}\,\mathrm{d}s + v(t)\sum_{\Gamma_{i0}}\int_{\Gamma_{i0}} \gamma_{i0}(\mathbf{w})\cdot v_{i0}\,\mathrm{d}s.
\end{aligned}
$$

This expression equals zero, since, with polyhedral cells, the v_{ij} are constant and hence, by assumption, the components of v_{ij} are in the span of the collocation weights $\omega_k^{ij}, k \leqslant n_{ij}$.

(d) and (e). For any $w \in C_0^\infty(\Omega)$, we form vector function $\mathbf{w}_k = (0, 0, \ldots, w, \ldots)$, where w is the kth component of $\mathbf{w}_k$. $\mathbf{w}_k$ is in $\mathbf{H}_0^1(\boldsymbol{\Omega})$. Suppose that $|v|_0^T = 0$ for $v \in C([0,1]; H^1(\Omega))$. Then, for each $t, 0 = (v, \nabla \cdot \mathbf{w}_k)_0 = -(D_k v, w)_0$. This holds for any k and for all w in $C_0^\infty(\Omega)$, which is dense in $L_2(\Omega)$, so $D_k v = 0$ in Ω, and $v(\cdot, t)$ is constant a.e in x. If $v \in C([0,T]; H_0^1(\Omega))$, then $v(\cdot, t) = 0$ a.e in x. If $v(\cdot, t)$ is a function $h(t)$ of t only, then, for $\mathbf{w}$ in $\mathbf{H}_0^1(\boldsymbol{\Omega}), (v, \nabla \cdot \mathbf{w})_0 = h(t) \int_\Gamma \gamma(\mathbf{w}) \cdot v \, \mathrm{d}s = 0$ by the divergence theorem, where Γ is the boundary of Ω, the trace map is γ and the outward unit normal is v. $\square$

We expect that $\Pi - \Pi_{mn}^r$ is close to a constant function for each t; it is in H for each t by assumption and construction. Thus, from part (e) of Lemma 1.7, $|\cdot|_0^T$ is a possible measure for convergence of Π_{mn}^r. Theorem 1.6 is not as strong as a similar result obtained for the stationary Stokes equations [22,25]. However, our experiments discussed in the next section suggest that a result stronger than that in Theorem 1.6 may hold when the basis functions are polynomials.

2. Error estimates for the algorithm using a polynomial basis

We have implemented this scheme for arbitrary problems with domains in $\mathbb{R}^2$ that can be partitioned into triangles or parallelograms (or both) using the methods described in [8,23,24]. On each cell, we use L_2-orthonormal bases (up to 66 basis functions) spanning polynomials of degree 10 or less [15] to provide approximations. Legendre polynomials are used as collocation weight functions on the interfaces.

If we employ a pth degree basis on each cell and we use $p+1$ collocation weight functions on the interfaces, then an approximation will be continuous, for the differences of the traces of an approximation on adjacent cells is a polynomial of degree p orthogonal to the first $p+1$ Legendre polynomials; such a difference must be zero. Thus our software allows us to generate the continuous functions of an h–p finite element approximation.

The divergence of a pth degree polynomial approximation on each cell is of degree $p-1$. Hence, if we force the divergence of such approximations to be orthogonal to polynomials of degree $p-1$ or less, the divergence must be zero. This argument holds for polynomial bases on cell decompositions of polyhedral domains in higher dimensions; continuity and the solenoidal condition can be enforced exactly.

However, in our experiments in $\mathbb{R}^2$, our best results occur when fewer collocation weight functions are used than are necessary for continuity; the question of enforcing continuity of approximations is extensively discussed in [8]. The optimal approximation to the pressure in the example we discuss in this section is achieved when the divergence of an approximation is required to be orthogonal to polynomials of degree $p-2$, one less than the degree necessary for the exact solenoidal requirement.

We use Gauss–Legendre quadrature to compute the integrals over the cells and interfaces. Subroutines from LINPACK and LAPACK provide the QR decomposition for $\mathscr{M}^\mathrm{T}$ and the eigenvalues and eigenvectors necessary to solve system (13) of ordinary differential equations.

We apply the results of Section 1 to derive error estimates in terms of p, the degree of the polynomial approximation on each cell, q, the maximum degree of the Legendre polynomials providing the weight functions on the interfaces on any trial, ρ, the maximum degree of the polynomials used to enforce the weak solenoidal condition and h, the maximum diameter of the cells in the partition of Ω. We use the same number $q+1$ of collocations on each boundary segment Γ_{ij}, so we set all n_{ij} to $q+1$; we revise notation containing collocation index $[n]$ by replacing $[n]$ with q. We use the same number $(p+1)(p+2)/2$ of basis functions for our pth degree basis on each cell; notation containing basis multi-index $[m]$ replaces $[m]$ by p. Finally, we use the same number $r=(\rho+1)(\rho+2)/2$ of basis functions for our ρth degree basis employed to achieve the weak solenoidal condition on each cell; notation r is replaced by ρ. Thus we denote approximation $\mathbf{u}^r_{mn}$ by symbol $\mathbf{u}^\rho_{pq}$.

The relevant error estimates for a polynomial implementation of these methods are given in [8], where two types of error estimates are discussed. The first is expressed in terms of the $H^k(\Omega)$ norm of the solution $(k>2)$ and is similar to conventional h–p estimates. The second assumes that the solution is very smooth and is expressed in terms of the semi-norm defined by the L_2-norm of high derivatives of the solution. We use these second estimates in our discussion here.

The trace constants C_{ij} are bounded by $c_1/h^{1/2}$, where c_1 is independent of h and depends only the smallest angle in any cell.

$$\|\mathscr{T}^{ij}_{q}\,(D_{v_{ij}}\mathbf{u})\|_{ij} \equiv \|\mathscr{T}^{ij}_{n_{ij}}(D_{v_{ij}}\mathbf{u})\|_{ij} \leqslant 0.66\times h^{q+1}(0.7(q+2))^{-(q+3/2)}\|(D_{v_{ij}}\mathbf{u})^{q+1}\|_{ij},$$

where $(D_{\mathbf{n}_{ij}}\mathbf{u})^{q+1}$ represents the $(q+1)$st tangential derivative of $D_{\mathbf{n}_{ij}}\mathbf{u}$ on Γ_{ij}.

The unit vectors v_{ij} normal to Γ_{ij} are constant for boundaries of parallelogram or triangular cells, so, using the linearity of $\mathscr{T}^{ij}_{n_{ij}} \equiv \mathscr{T}^{ij}_{q}$, it follows that

$$\|\mathscr{T}^{ij}_{q}(\gamma_{ij}(\Pi)v^{\cdot}_{ij})\|_{ij} = \|\mathscr{T}^{ij}_{q}(\gamma_{ij}(\Pi))\|_{ij}.$$

From [8], $\|\mathscr{T}^{ij}_{q}(\gamma_{ij}(\Pi))\|_{ij} \leqslant 0.66\times h^{q+1}(0.7(q+2))^{-(q+3/2)}\|(\gamma_{ij}(\Pi))^{q+1}\|_{ij}$.

For our cells, we have for $\mathbf{v}\in\mathbf{H}^{p+2}(\mathbf{\Omega})$ (and for $h\leqslant 3$ and $p\geqslant 2$),

$$\|\mathbf{Q}_p\mathbf{v}\|_H \leqslant h^p(0.5p)^{-p}[|\mathbf{v}|_{p+1}+|\mathbf{v}|_{p+2}], \text{ where, for example, } |\mathbf{v}|^2_{p+1} = \sum_{|\alpha|=p+1}\|D^\alpha\mathbf{v}\|^2_0.$$

From [8], for $\Pi\in H^{\rho+2}(\Omega)$,

$$\|\mathscr{T}_\rho(\Pi)\|_0 \leqslant 0.2h^{\rho+1}(0.5(\rho+1))^{-(\rho+1)}[|\Pi|_{\rho+1}+|\Pi|_{\rho+2}].$$

If we apply Theorem 1.2 and use these estimates with q, p and ρ replacing $[n], [m]$ and r, and $K(n,r){:=}K(q,\rho)$, we get

$$\begin{aligned}
&\|\mathbf{u}(\tau)-\mathbf{u}^\rho_{pq}(\tau)\|^2_0 + c\int_0^\tau \|\mathbf{u}-\mathbf{u}^\rho_{pq}\|^2_H(t)\,\mathrm{d}t\\
&\quad\leqslant 2K(q,\rho)^2(3\|\mathbf{Q}_p\mathbf{u}^0\|^2_H + 2\|\mathbf{Q}_p\mathbf{u}(\tau)\|^2_H)\\
&\qquad+\frac{20}{3c}\int_0^\tau [K(q,\rho)^2\{\|\mathbf{Q}_p\mathbf{u}\|^2_H + \|\mathbf{Q}_p\mathbf{u}'\|^2_H\} + K\|\mathscr{T}_\rho(\Pi)\|^2_0]\,\mathrm{d}t\\
&\qquad+Nn_f^2\sup\{C^2_{ij}\}\frac{20}{3c}\int_0^\tau (\sup\{\|\mathscr{T}^{ij}_{n_{ij}}(\gamma_{ij}(\Pi)v^{\cdot}_{ij})\|^2_{ij}\} + \sup\{\|\mathscr{T}^{ij}_{n_{ij}}(D_{\mathbf{n}_{ij}}\mathbf{u})\|^2_{ij}\})\,\mathrm{d}t\\
&\quad\leqslant (1+(1/\mu)[8c_1^2/h + 2(\rho+1)(\rho+2)/2])[h^p(0.5p)^{-p}]^2
\end{aligned}$$

p	$\sqrt{\frac{1}{\mu_1}}$ for a triangle	$\sqrt{\frac{1}{\mu_1}}$ for a square
3	13	7.3
4	39	13.3
5	55	26
6	80	40
7	104	92
8	134	92
9	167	270
10	207	260

Fig. 1. Values for $\sqrt{1/\mu_1}$ for the unit triangle and square as a function of p; $q = p - 3$ and $\rho = p - 1$.

$$
\begin{aligned}
&\times \Bigg\{ 12[|\mathbf{u}^0|_{p+1}^2 + |\mathbf{u}^0|_{p+2}^2 + |\mathbf{u}(\tau)|_{p+1}^2 + |\mathbf{u}(\tau)|_{p+2}^2] \\
&+ \frac{14}{c}\int_0^\tau |\mathbf{u}(t)|_{p+1}^2 + |\mathbf{u}(t)|_{p+2}^2 + |\mathbf{u}'(t)|_{p+1}^2 + |\mathbf{u}'(t)|_{p+2}^2\,\mathrm{d}t \Bigg\} \\
&+ \frac{1}{c}[h^{\rho+1}(0.5(\rho+1))^{-(\rho+1)}]^2 \int_0^\tau [|\Pi(t)|_{\rho+1}^2 + |\Pi(t)|_{\rho+2}^2]\,\mathrm{d}t \\
&+ 47\frac{N}{h}c_1^2[h^{q+1}(0.7(q+2))^{-(q+3/2)}]^2 \int_0^\tau (\sup\{\|(\gamma_{ij}(\Pi))^{q+1}\|_{ij}^2\} \\
&+ \sup\{\|(D_{\mathbf{v}_{ij}}\mathbf{u})^{q+1}\|_{ij}^2\})\,\mathrm{d}t.
\end{aligned} \tag{29}
$$

Values for $1/\mu$ are discussed extensively in [8] and (specifically for the Stokes equations) in [25]. It is shown in [25] that it suffices to calculate these values for representative cells of a partition; the values then hold for any domain that is the union of such cells. Fig. 1 gives the relevant values for $\sqrt{1/\mu_1}$ for triangular or square cells with side length equal to 1. When we calculate $\sqrt{1/\mu_h}$ for cells with side h, we get $\sqrt{1/\mu_h} \leqslant h^{-1}\sqrt{1/\mu_1}$.

If we are subdividing the unit square into square cells of side h, the number of cells $N \cong 1/h^2$. Due to the resulting decrease in the size of Γ_{ij}, we might expect $\|(D_{\mathbf{n}_{ij}}\mathbf{u})^{q+1}\|_{ij}$ and $\|(\gamma_{ij}(\Pi))^{q+1}\|_{ij}$ to decrease by a factor $h^{1/2}$. Then the h-dependency of the last error estimate term containing the normal derivative of the solution and $\gamma_{ij}(\Pi)$ on the interfaces would be $h^{-2}h^{-1}h^{(2q+2)}h^1 = h^{2q}$. We use this observation in the following more concise version of (29):

$$
\begin{aligned}
&\|\mathbf{u}(\tau) - \mathbf{u}_{pq}^\rho(\tau)\|_0 + \left[c\int_0^\tau \|\mathbf{u} - \mathbf{u}_{pq}^\rho\|_H^2(t)\,\mathrm{d}t\right]^{1/2} \\
&\leqslant (c_2h^{-1/2} + c_3\rho)(h^p(0.5p)^{-p})\mathscr{E}_1(\mathbf{u}^0, \mathbf{u}, \mathbf{u}') \\
&+ h^{\rho+1}(0.5(\rho+1))^{-(\rho+1)}\mathscr{E}_2(\Pi) + h^q(0.7(q+2))^{-(q+3/2)}\mathscr{E}_3(\mathbf{u}, \Pi),
\end{aligned} \tag{30}
$$

where the $\mathscr{E}_i(\cdot)$ represent the dependency on $1/\mu, \mathbf{u}^0, \mathbf{u}, \mathbf{u}'$ and Π given in (29).

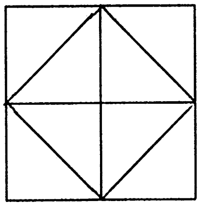

Fig. 2. Cell structure.

3. A numerical example

Our sample problem for numerical tests is to approximate the solution to the nonstationary Stokes equations defined by using solenoidal vector field

$$\mathbf{u} = (\exp(t)\sin^2(\pi x)\sin(2\pi y), -\exp(t)\sin(2\pi x)\sin^2(\pi y))$$

with pressure $\Pi = \sin(xyt)$. The domain is the unit square; the cells partition it into the 8 triangles shown in Fig. 2.

Tests were made to obtain approximation errors for various values of p, q, and ρ. Although the theory only gives estimates for the *integral* of the square of the **H**-norm of the velocity error and the $|\cdot|_{0[m][n]}$ semi-norm of the pressure error described in Definition 1.5, we calculated both the $\mathbf{L}_2$ and **H** norms of the velocity errors and $\|\mathrm{grad}(\Pi^r_{mn} - \Pi)\|_0$ at time τ to see if, in fact, convergence appears to hold in these stronger measures. The difference between the true solution and the approximation was calculated on a uniform 41×41 grid; the squares of the $\mathbf{L}_2$ and **H** errors are evaluated using ELLPACK's technique [21] of computing the average of the squares of the differences plus (for the **H** errors) the squares of the differences of the derivatives.

In our experiments, although the velocity errors were essentially the same with $\rho = p - 1$(where the exact solenoidal condition was enforced) and $\rho = p - 2$, optimal results for the approximate pressure were obtained when $\rho = p - 2$. As for weak continuity, the best results were obtained when $q = p - 3$ or $q = p - 4$. With such substitutions for ρ and q, disregarding $1/\mu$, the theoretical p dependency of the velocity error in (30) is of form $Cp(0.5p)^{-p}$ with C depending on various norms and semi-norms of the solution $\langle \mathbf{u}, \Pi \rangle$. We plot the $\mathbf{L}_2$ and **H** errors of the approximation in Fig. 3 for p ranging from 3 to 10.

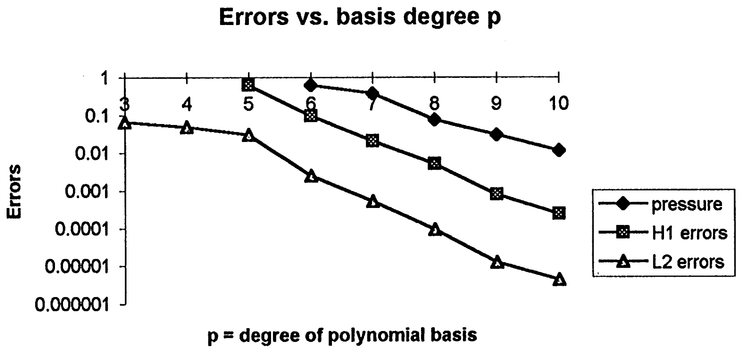

Fig. 3. Pressure and velocity errors vs. p.

The theoretical error contains norms of p (or more) derivatives of $\mathbf{u}$ which contain the factor π^p. If we include this in our theoretical estimate, we obtain (velocity error) $\leqslant Cp(0.159p)^{-p}$.

The empirical regression formulae using error model 'error $= Cp(Kp)^{-p}$' are

$$\mathbf{L}_2\text{-error} \cong 0.027p(0.318p)^{-p}$$

and

$$\mathbf{H}\text{-error} \cong 0.411p(0.276p)^{-p}.$$

The empirical regression approximation for the pressure error (using the model error $\leqslant C(Kp)^{-p}$) is

$$\|\text{grad}(\Pi^r_{mn} - \Pi)\|_0 \cong 0.164(0.13p)^{-p}.$$

The gradient of the pressure error is also shown in Fig. 3.

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